Mathematics

Algebra II

DETROIT PUBLIC SCHOOLS COMMUNITY DISTRICT

Letter to Families from the DPSCD Office of Mathematics

Dear DPSCD Families,

The Office of Mathematics is partnering with families to support Distance Learning while students are home. We empower you to utilize the resources provided to foster a deeper understanding of grade-level mathematics.

In this packet, you will find links to videos, links to online practice, and pencil-and-paper practice problems. The Table of Contents shows day-by-day lessons from April 14th to June 19th. We encourage you to take every advantage of the material in this packet.

Daily lesson guidance can be found in the table of contents below. Each day has been designed to provide you access to materials from Khan Academy and the academic packet. Each lesson has this structure:

Watch: Khan Academy (if internet access is available)	**Practice: Khan Academy (if internet access is available)**	**Pencil & Paper Practice: Academic Packet**
Watch and take notes on the lesson video on Khan Academy	Complete the practice exercises on Khan Academy	Complete the pencil and paper practice.

If one-on-one, live support is required, please feel free to call the **Homework Hotline** at **1-833-466-3978**. Please check the Homework Hotline page for operating hours. We have DPSCD mathematics teachers standing by and are ready to assist.

We appreciate your continued dedication, support and partnership with Detroit Public Schools Community District and with your assistance we can press forward with our priority: Outstanding Achievement. Be safe. Be well!

Deputy Executive Director of K-12 Mathematics

Important Links and Information

Clever

Students access Clever by visiting [www.clever.com/in/dpscd.](http://www.clever.com/in/dpscd)

What are my username and password for Clever?
Students access Clever using their DPSCD login credentials. Usernames and passwords follow this structure:

<u>Username</u>: studentID@thedps.org
Ex. If Aretha Franklin is a DPSCD student with a student ID of 018765 her username would be 018765@thedps.org.

<u>Password</u>:
First letter of first name in upper case
First letter of last name in lower case
2-digit month of birth
2-digit year of birth
01 (male) or 02 (female)
For example: If Aretha Franklin's birthday is March 25, 1998, her password and password would be Af039802.

Accessing Khan Academy

To access Khan Academy, visit [www.clever.com/in/dpscd.](http://www.clever.com/in/dpscd) Once logged into Clever, select the Khan Academy button:

Khan Academy

Accessing Your CPM eBook

Students can access their CPM eBook in two ways:

Option 1: Access the eBook through <u>Clever</u>

1. Visit [www.clever.com/in/dpscd.](www.clever.com/in/dpscd) Login using your DPSCD credentials above.
2. Click on the CPM icon:

Option 2: Visit http://open-ebooks.cpm.org/

1. Visit the website listed above.
2. Click "I agree"
3. Select the CPM Algebra II eBook:

CC Algebra 2

Desmos Online Graphing Calculator

Access to a free online graphing and scientific calculator can be found at https://www.desmos.com/calculator.

Table of Contents

In the following table, you will find the table of contents and schedule for the week of April 13, 2020 through the week of June 15, 2020.

Week	Date	Topic	Watch (10 minutes)	Online Practice (10 minutes)	Pencil & Paper Practice (25 minutes)
Week of 04/13-04/17	Monday, 4/13/20 Day 1	Holiday	N/A	N/A	N/A
	Day 2	Chapter 2.1.1 – 2.1.5: Transformations of x^2 (Day 1)	Quadratic Formula Finding Features of Quadratic Functions	Practice: Quadratic Formula Practice: Finding Features of Quadratic Functions	Problems 1 - 8
	Day 3	Chapter 2.1.1 – 2.1.5: Transformations of x^2 (Day 2)	Quadratic Word Problem	Practice: Quadratic Word Problems	Problems 9 - 15
	Day 4	Chapter 2.2.1 – 2.2.3: Transforming Parent Graphs (Day 1)	Intro to Parabola Transformations	Practice: Shifting Parabolas	Problems 1 - 6

			Shifting Parabolas Scaling/Reflecting Parabolas	Practice: Scale/Reflect Parabolas	
	Day 5	Chapter 2.2.1 – 2.2.3: Transforming Parent Graphs (Day 2)	Identifying Function Transformations	Practice: Identifying Transformations	Problems 7 - 14
Week of 4/20-4/24	Day 1	Chapter 2.2.3: More on Completing the Square (Day 1)	Completing the Square Equations of Circles	Practice: Completing the square	Problems 1 - 12
	Day 2	Chapter 2 SAT Prep			Problems 1 - 10

	Day 3	Chapter 3.1.1 – 3.1.2: Equivalence (Day 1)	Simplifying Exponents Simplifying Exponential Expressions Equivalent Expressions	Practice: Simplifying Exponents Practice: Equivalent Expressions	Problems 1 - 7
	Day 4	Chapter 3.1.1 – 3.1.2: Equivalence (Day 2)	Arithmetic sequences Geometric Sequences	Practice: Arithmetic Sequence Practice: Geometric Sequence	Problems 8 - 15
	Day 5	Chapter 3.2.2: Simplifying Expressions (Day 1)	Simplifying Rational Expressions	Practice: Simplifying Rational Expressions	Problems 1 - 10

Week of 4/27-05/01	Day 1	Chapter 3.2.2: Simplifying Expressions (Day 2)	Simplifying Rational Expressions Binomials Simplifying Rational Expressions Binomials	Practice: Simplifying Rational Expressions	Problems 11 - 21
	Day 2	Chapter 3.2.3: Multiplying and Dividing Rational Expressions (Day 1)	Multiplying Rational Expressions	Practice: Multiplying Rational Expressions	Problems 1 - 7
	Day 3	Chapter 3.2.3: Multiplying and Dividing Rational Expressions (Day 2)	Dividing Rational Expressions	Practice: Dividing Rational Expressions	Problems 8 - 15
	Day 4	Chapter 3.2.4: Adding and Subtracting Rational Expressions (Day 1)	Adding & Subtracting Rational Expressions	Practice: Adding and Subtracting Rational Expressions	Problems 1 - 10

			Summary (+ & -) Rational Expressions		
	Day 5	Chapter 3.2.4: Adding and Subtracting Rational Expressions (Day 2)	Adding & Subtracting Rational Expressions	Practice: Adding & Subtracting Rational Expression	Problems 11 - 20
Week of 05/04-05/08	Day 1	Chapter 3 SAT Prep	SAT Practice	SAT Practice	Problems 1 - 10
	Day 2	Chapter 4.1.1 – 4.1.4: Solving Systems of Equations (Day 1)	Solving Systems of Equations by graphing Solving SOE by Substitution	Practice: Solving SOE by graphing Practice: Solving SOE by Substitution	Problems 1 - 6

			Solving SOE by elimination	Practice: Solving SOE by Elimination (linear combination)	
	Day 3	Chapter 4.1.1 – 4.1.4: Solving Systems of Equations (Day 2)	Using a graph to evaluate (solve) a function	Practice: Using a graph to evaluate a function	Problems 7 - 10
	Day 4	Chapter 4.1.1 – 4.1.4: Solving Systems of Equations (Day 3)	Solving Rational Expression Equations Solving Radical Equations	Practice: Solving Rational Equations Practice: Solving Radical Equations	Problems 11 - 14

	Day 5	Chapter 4.1.1 – 4.1.4: Solving Systems of Equations (Day 4)	Solving nonlinear Systems by graphing Solve nonlinear SOE by graphing		Problems 15 - 21
Week of 05/11-05/15	Day 1	Chapter 4.2.1 – 4.2.4: Inequalities (Day 1)	Solving Systems of Inequalities by graphing	Practice: Solving Systems of Inequalities by graphing	Problems 1 - 3
	Day 2	Chapter 4.2.1 – 4.2.4: Inequalities (Day 2)	Solving Systems of Inequalities by Graphing	Practice: Solving Systems of Inequalities by graphing	Problems 4 - 6

	Day 3	Chapter 4.2.1 – 4.2.4: Inequalities (Day 3)	Writing Systems of Inequalities	Practice: Writing Systems of Inequalities	Problems 7 - 10
	Day 4	Chapter 4 SAT Prep			Problems 1 - 10
	Day 5	Chapter 5.1.1 – 5.1.3: Inverses (Day 1)	Finding Inverse Functions	Practice: Finding Inverse Functions	Problems 1 - 10
Week of 05/18-05/22	Day 1	Chapter 5.1.1 – 5.1.3 (Day 2)	Graphing Inverse Functions	Practice: Graphing Inverse Functions	Problems 11 - 15
	Day 2	Chapter 5.1.1 – 5.1.3 (Day 3)	Verifying Function Inverses by Compositions	Practice: Verifying Inverse Functions by Composition	Problems 16 - 21

			Summary of Verifying Inverse Functions by Composition		
	Day 3	Chapter 5.2.1 – 5.2.4: Logarithms (Day 1)	Relationship between Exponential form and Logarithmic form	Practice: Exponential form to Logarithmic form and back	Problems 1 - 10
	Day 4	Chapter 5.2.1 – 5.2.4: Logarithms (Day 2)	Solving Exponential Equations with a Log	Practice: Solving Exponential Equations with a Log	Problems 11 - 20
	Day 5	Chapter 5.2.1 – 5.2.4: Logarithms (Day 3)	Graphing Log Functions	Practice: Graphing Log Equations	Problems 21 - 24
Week of	Day 1	Holiday	N/A	N/A	N/A
	Day 2	Chapter 5 SAT Prep	SAT Practice	SAT Practice	Problems 1 - 10

05/25-05/29	Day 3	Chapter 6.2.1 – 6.2.4: Solving with Logarithms (Day 1)	Exponential equations with logs	Practice: Logs and Exponential Relationships	Problems 1 - 10
	Day 4	Chapter 6.2.1 – 6.2.4: Solving with Logarithms (Day 2)	Intro to Log Properties	Practice: Properties of Logs	Problems 11 - 15
	Day 5	Chapter 6.2.1 – 6.2.4: Solving with Logarithms (Day 3)	Intro to Logs	Practice: Evaluate Logs Practice: Evaluate Log (advanced)	Problems 16 - 25
Week of	Day 1	Chapter 6 SAT Prep	SAT Practice	SAT Practice	Problems 1 - 10

06/01-06/05	Day 2	Chapter 7.1.1 – 7.1.7: Trigonometric Functions (Day 1)	Graph of sin x Graph of tan x		Problems 1 - 3
	Day 3	Chapter 7.1.1 – 7.1.3: Trigonometric Functions (Day 2)	Radians and Degrees	Practice Degrees and Radians	Problems 4 - 9
	Day 4	Chapter 7.1.1 – 7.1.3: Trigonometric Functions (Day 3)	Convert Degrees to Radians Convert Radians to Degrees	Practice: Degree to Radian Practice: Unit Circle with Radians	Problems 10 - 15

	Day 5	Chapter 7.2.1 – 7.2.4: Transforming Trigonometric Functions (Day 1)	Features of Sine Functions	Practice: Amp Problems Practice: Period Problems	Problems 1 - 4
Week of 06/08-06/12	Day 1	Chapter 7.2.1 – 7.2.4: Transforming Trigonometric Functions (Day 2)	Sine Features from their Equations	Practice: Amp Problems from Equations Practice: Period Problems from Equations	Problems 5 - 10
	Day 2	Chapter 7.2.1 – 7.2.4: Transforming Trigonometric Functions (Day 3)	Transform Sin Graphs (vertical/reflection)	Practice: Graph Sin Functions	Problems 11 - 17

			Transform Sin Graph (horizontal stretch)	Practice: Phase Shifts	
	Day 3	Chapter 7 SAT Prep	SAT Practice	SAT Practice	Problems 1 - 10
	Day 4	Chapter 8.1.1 – 8.1.3: Polynomials (Day 1)	Intro to Polynomials	Practice: Classifying Polynomials	Problems 1 - 7
	Day 5	Chapter 8.1.1 – 8.1.3: Polynomials (Day 2)	Graphs of polynomials	Practice: Zeros of Polynomials	Problems 8 - 13
Week of 06/15-06/19	Day 1	Chapter 8.2.1 – 8.2.3: Complex Numbers (Day 1)	Intro to Complex	Practice: Parts of Complex	Problems 1 - 6

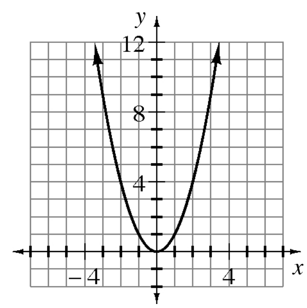
			Adding Complex Multiplying Complex	Practice: Adding Complex Practice: Multiplying Complex	
	Day 2	Chapter 8.2.1 – 8.2.3: Complex Numbers (Day 2)	Graphs of polynomials	Practice: Zeros of Polynomials	Problems 7 - 9
	Day 3	Chapter 8.3.1 – 8.3.3: Factoring Polynomial Functions (Day 1)	Dividing Polynomials	Practice: Dividing Polynomials	Problems 1 - 3
	Day 4	Chapter 8.3.1 – 8.3.3: Factoring Polynomial Functions (Day 2)	Factoring using Area Models	Practice: Factoring Polynomials	Problems 4 - 7

	Day 5	Chapter 8 SAT Prep	SAT Practice	SAT Practice	Problems 1 - 10

TRANSFORMATIONS OF $f(x) = x^2$ 2.1.1 – 2.1.5

In order for the students to be proficient in modeling data or contextual relationships, they must easily recognize and manipulate graphs of various functions. Students investigate the general equation for a family of quadratic functions, discovering ways to shift and change the graphs. Additionally, they learn how to quickly graph a quadratic function when it is written in graphing form. For further information see the Math Notes box in Lesson 2.1.4.

Example 1

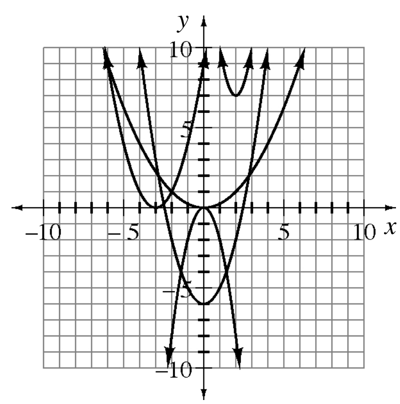


The graph of $f(x) = x^2$ is shown at right. For each new function listed below, explain how the new graph would differ from this original graph.

$g(x) = -2x^2$ $\quad h(x) = (x+3)^2$ $\quad j(x) = x^2 - 6$

$k(x) = \frac{1}{4}x^2$ $\quad l(x) = 3(x-2)^2 + 7$

Every function listed above has something in common: they all have 2 as the highest power of x. This means that all of these functions are quadratic functions, and all will form a parabola when graphed. The only differences will be in the direction of opening (up or down), the size (compressed or stretched), and/or the location of the vertex.



The "–2" in $g(x) = -2x^2$ does two things to the parabola. The negative sign changes the parabola's direction so that it will open downward. The "2" stretches the graph making it appear skinnier. The graph of $h(x) = (x + 3)^2$ will have the same shape as $f(x) = x^2$, open upward, and have a new location: it will move to the **left** 3 units. The graph of $j(x) = x^2 - 6$ will also have the same shape as $f(x) = x^2$, open upward and be shifted **down** 6 units.

The function $k(x) = \frac{1}{4}x^2$ does not move, still opens upward, but the $\frac{1}{4}$ will compress the parabola, making it appear "fatter." The last function, $l(x) = 3(x - 2)^2 + 7$, combines all of these changes into one graph. The "3" causes the graph to be skinnier and open upward, the "–2" causes it to shift to the **right** 2 units, and the "+ 7" causes the graph to shift **up** 7 units. All these graphs are shown above. Match the equation with the correct parabola.

Example 2

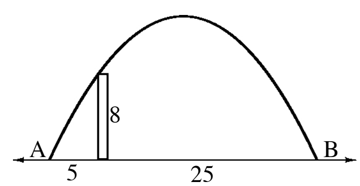
For each of the quadratic equations below, where is the vertex?

$$f(x) = -2(x+4)^2 + 7 \qquad g(x) = 5(x-8)^2 \qquad h(x) = \tfrac{3}{5}x^2 - \tfrac{2}{5}$$

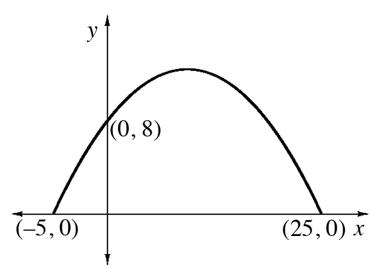
For a quadratic equation, the vertex is the **locator point**. It gives you a starting point for graphing the parabola quickly. The vertex for the quadratic equation $f(x) = a(x-h)^2 + k$ is the point (h, k). For $f(x) = -2(x + 4)^2 + 7$ the vertex is $(-4, 7)$. Since $g(x) = 5(x - 8)^2$ can also be written $g(x) = 5(x - 8)^2 + 0$, the vertex is $(8, 0)$. We can rewrite $h(x) = \frac{3}{5}x^2 - \frac{2}{5}$ as $h(x) = \frac{3}{5}(x-0)^2 - \frac{2}{5}$ to see that its vertex is $\left(0, \frac{2}{5}\right)$.

Example 3

In a neighborhood water balloon battle, Dudley has developed a winning strategy. He has his home base situated five feet behind an eight-foot fence. 25 feet away on the other side of the fence is his nemesis' camp. Dudley uses a water balloon launcher, and shoots his balloons so that they just miss the fence and land in his opponent's camp. Write an equation that, when graphed, will model the trajectory (path) of the water balloon.



As with many problems, it is most helpful to first draw a sketch of the situation. The parabola shows the path the balloon will take, starting five feet away from the fence (point A) and landing 25 feet past the fence (point B).



There are different ways to set up axes for this problem, and depending where you put them, your answer might be different. Here, the y-axis will be at the fence. With the axes in place, we label any coordinates we know. This now shows all of the information we have from the problem description. If we can find the coordinates of the vertex (highest point) of this parabola, we will be able to write the equation of it in graphing form. Parabolas are symmetric, therefore the vertex will be half-way between the two x-intercepts. The total distance between points A and B is 30 units, so half is 15. Fifteen units from point A is the point $(10, 0)$. We know the equation will be in the form $y = a(x-10)^2 + k$, with $a < 0$. Also, k must be greater than eight since the vertex is higher than the y-intercept of $(0, 8)$. The parabola passes through the points $(0, 8)$, $(-5, 0)$ and $(25, 0)$. We will use these points in the equation we have so far and see what else we can find.
Using the point $(0, 8)$ we can substitute the x- and y-values in to the equation and write:

$$8 = a(0-10)^2 + k$$
or
$$8 = 100a + k$$

This equation has two variables, which means we need another (different) equation with a and k to be able to solve for them. Using the point $(-5, 0)$:

$$0 = a(-5-10)^2 + k$$
$$\text{or}$$
$$0 = 225a + k$$

Begin solving by subtracting the second equation from the first:

$$\begin{array}{r} 8 = 100a + k \\ -(0 = 225a + k) \\ \hline 8 = -125a \\ a = -\frac{8}{125} \end{array}$$

Substitute this value for the variable a back into one of the two equations above to find k.

$$8 = 100\left(-\tfrac{8}{125}\right) + k$$
$$8 = -\tfrac{32}{5} + k$$
$$k = \tfrac{72}{5}$$

The equation for the path of a water balloon is $y = -\frac{8}{125}(x-10)^2 + \frac{72}{5}$. You should graph this on your graphing calculator to check.

Problems

Find the x- and y-intercepts of each of the following quadratic equations.

1. $y = x^2 + 4x + 3$
2. $y = x^2 + 5x - 6$
3. $y = 2x^2 - 7x - 4$
4. $y = -3x^2 - 10x + 8$
5. $y = 16x^2 - 25$
6. $y = 6x - 12$

Find the error in each of the following solutions. Then find the correct solution to the problem.

7. Solve for x if $3x^2 + 6x + 1 = 0$.

   $a = 3, b = 6, c = 1$

   $$\begin{aligned} x &= \tfrac{6 \pm \sqrt{6^2 - 4(3)(1)}}{2(3)} \\ &= \tfrac{6 \pm \sqrt{36-12}}{6} \\ &= \tfrac{6 \pm \sqrt{24}}{6} \\ &= \tfrac{6 \pm 2\sqrt{6}}{6} \\ &= \tfrac{3 \pm \sqrt{6}}{3} \end{aligned}$$

8. Solve for x if $-2x^2 + 7x + 5 = 0$

   $a = -2, b = 7, c = 5$

   $$\begin{aligned} x &= \tfrac{-7 \pm \sqrt{7^2 - 4(-2)(5)}}{2(-2)} \\ &= \tfrac{-7 \pm \sqrt{49-40}}{-4} \\ &= \tfrac{-7 \pm 3}{-4} \end{aligned}$$

   $x = \frac{-4}{-4} = 1$ or $x = \frac{-10}{-4} = 2.5$

Find the vertex of each of the following parabolas by "averaging the x-intercepts" or "completing the square." Then write each equation in graphing form, and sketch the graph.

9. $y=-2x^2+4x+1$

10. $y=x^2+10x+19$

11. $y=(x+7)(x-3)$

12. $y=2(x+6)^2-1$

For each situation, write an appropriate equation that will model the situation effectively.

13. When Twinkle Toes Tony kicked a football, it landed 100 feet from where he kicked it. It also reached a height of 125 feet. Write an equation that, when graphed, will model the path of the ball from the moment it was kicked until it first touched the ground.

14. When some software companies develop software, they do it with "planned obsolescence" in mind. This means that they plan on the sale of the software to rise, hit a point of maximum sales, then drop and eventually stop when they release a newer version of the software. Suppose the curve showing the number of sales over time is parabolic and that the company plans on the "life span" of its product to be six months, with maximum sales reaching 1.5 million units. Write an equation that best fits this data.

15. A new skateboarder's ramp just arrived at Bungey's Family Fun Center. A cross-sectional view shows that the shape is parabolic. The sides are 12 feet high and 15 feet apart. Write an equation that, when graphed, will show the cross section of this ramp.

Answers

1. x-intercepts: $(-1, 0)$, $(-3, 0)$, y-intercept: $(0, 3)$

2. x-intercepts: $(-6, 0)$, $(1, 0)$, y-intercept: $(0, -6)$

3. x-intercepts: $(-0.5, 0)$, $(4, 0)$, y-intercept: $(0, -4)$

4. x-intercepts: $(-4, 0)$, $\left(\frac{2}{3}, 0\right)$, y-intercept: $(0, 8)$

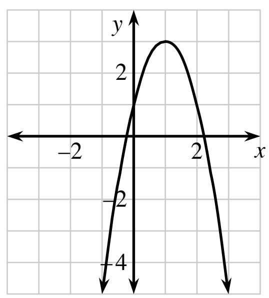
5. x-intercepts: $\left(-\frac{5}{4}, 0\right)$, $\left(\frac{5}{4}, 0\right)$, y-intercept: $(0, -25)$

6. x-intercept: $(2, 0)$, y-intercept: $(0, -12)$

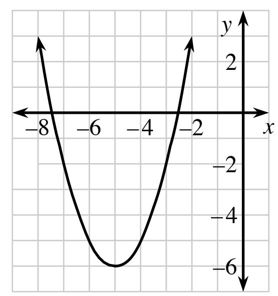
7. The formula starts with "$-b$;" the negative sign is left off. $x = \frac{-3 \pm \sqrt{6}}{3}$

8. Under the radical, "$-4ac$" should equal + 40. $x = \frac{-7 \pm \sqrt{89}}{-4}$

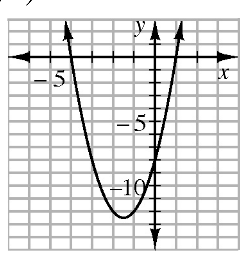
9. $y = -2(x-1)^2 + 3$
   x-intercepts: $\left(\frac{2 \pm \sqrt{6}}{2}, 0\right)$
   vertex is $(1, 3)$



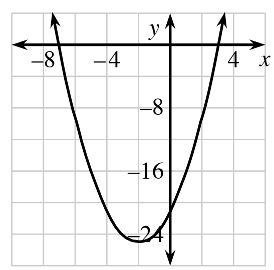
10. $y = (x+5)^2 - 6$
    x-intercepts: $\left(-5 \pm \sqrt{6}, 0\right)$
    vertex is $(-5, -6)$



11. $y = (x+2)^2 - 25$
    x-intercepts: $(3, 0)$, $(-7, 0)$
    vertex is $(-2, -25)$



12. $y = 2(x+6)^2 - 1$
    x-intercepts: $\left(\frac{-12 \pm \sqrt{2}}{2}, 0\right)$
    vertex is $(-6, -1)$



13. Placing the start of the kick at the origin gives $y = -0.05x(x - 100)$.

14. Let the x-axis be the number of months, and the y-axis be the number of sales in millions. Placing the origin at the beginning of sales, we can use $y = -\frac{1}{6}(x-3)^2 + 1.5$.

15. Placing the lowest point of the ramp at the origin gives $y = \frac{48}{225}x^2$.

Students will generalize what they have learned about transforming the graph of $f(x) = x^2$ to change the shape and position of the graphs of several other functions. The students start with the simplest form of each function's graph, which is called the "parent graph." Students use $y = x^3$, $y = \frac{1}{x}$, $y = \sqrt{x}$, and $y = b^x$ as the equations for parent graphs, and what they learn while studying these graphs will apply to all functions. They also learn to apply their knowledge to non-functions. For further information see the Math Notes boxes in Lessons 2.2.2 and 2.2.3.

Example 1

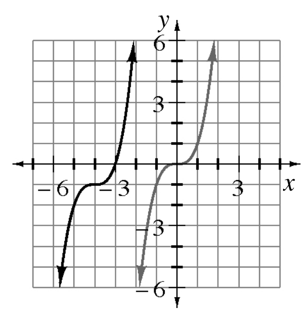
For each of the following equations, state the parent equation, and use it to graph each equation as a transformation of its parent equation.

$$y = (x+4)^3 - 1 \qquad\qquad y = -\frac{1}{x}$$

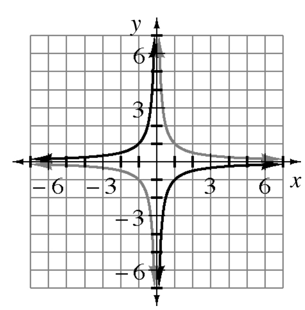
$$y = 3\sqrt{x-2} \qquad\qquad y = 3^x - 6$$

For each of these equations, we will graph both it and its parent on the same set of axes to help display the change and movement.

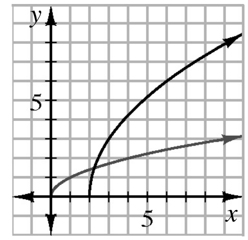
The first equation is a cubic (the term given to a polynomial with 3 as the highest power of x), thus its parent is $y = x^3$. The given equation will have the same shape as $y = x^3$, but it will be shifted to the left 4 units (from the "+ 4" within the parentheses), and down one unit (from the "–1"). The new graph is the darker curve shown on the graph at right. Notice that the point $(0, 0)$ on the original graph has shifted left 4, and down 1, and now is at $(–4, –1)$. This point is known as a **locator point**. It is a key point of the graph, and graphing its position helps us to graph the rest of the curve.



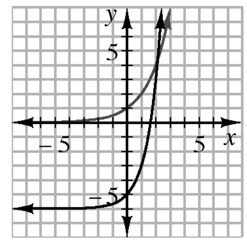
The second curve, $y = -\frac{1}{x}$, has had only one change from the parent graph $y = \frac{1}{x}$: the negative sign. Just as a negative at the front of $f(x) = x^2$ would flip this graph upside down, the negative sign here "flips" the parts of the parent graph across the x-axis. The lighter graph shown at right is the parent $y = \frac{1}{x}$, and the darker graph is $y = -\frac{1}{x}$.



In the equation $y = 3\sqrt{x-2}$ the radical is multiplied by 3, hence the transformed graph will grow vertically more quickly than the parent graph $y = \sqrt{x}$. It is also shifted to the right 2 units because of the "–2" under the radical sign. The new graph is the darker curve on the graph shown at right. Notice that the point (0, 0) on the original graph (the locator point) has shifted right 2 units.



This last graph is an exponential function. The parent graph, $y = b^x$, changes in steepness as b changes. The larger b is, the quicker the graph rises, making a steeper graph. With $y = 3^x - 6$, the graph is a bit steeper than $y = 2^x$, often thought of as the simplest exponential function, but also shifted down 6 units. The lighter graph is $y = 2^x$, while the darker graph is $y = 3^x - 6$.

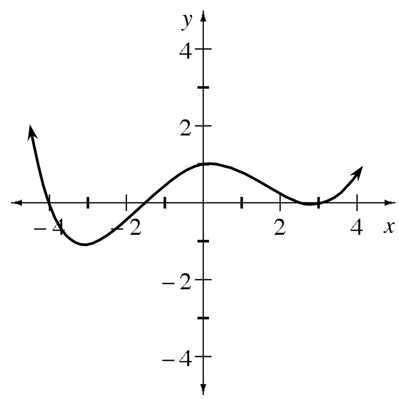


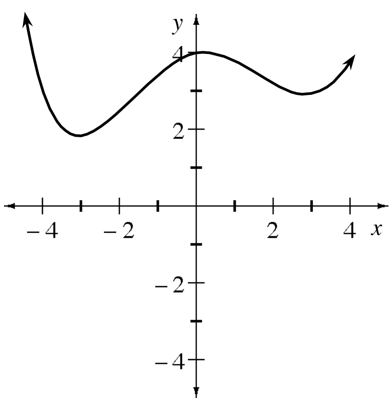
Example 2

Suppose $f(x)$ is shown at right. From all you have learned about changing the graphs of functions:

a. Graph $f(x) + 3$. b. Graph $f(x) - 2$. c. Graph $f(x - 1)$.

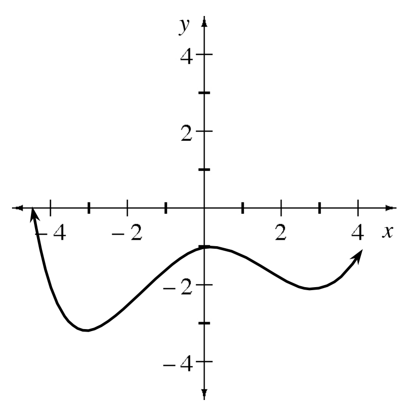
d. Graph $f(x + 3)$. e. Graph $3f(x)$. f. Graph $\frac{1}{2}f(x)$.





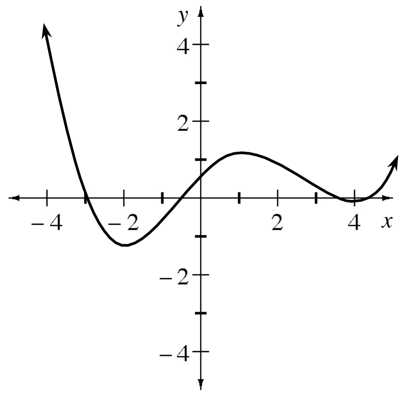
Each time we alter the equation slightly, the graph is changed. Even though we have no idea what the equation of this function is, we can still shift it on the coordinate grid. Remember that $f(x)$ represents the range or y-values. Therefore, in part (a), $f(x) + 3$ says "the y-values, plus 3." Adding three to all the y-values will shift the graph up three units. This is shown at left. Notice that the shape of the graph is identical to the original, just shifted up three units. Check this by comparing the y-intercepts.

If $f(x) + 3$ shifts the graph up three units, then $f(x) - 2$ will shift the graph down two units. This graph is shown at right. Again, compare the y-intercept on the graph at left to the original. (Note: Using the y-intercept or the x-intercepts to help you graph is an effective way to create a graph.)

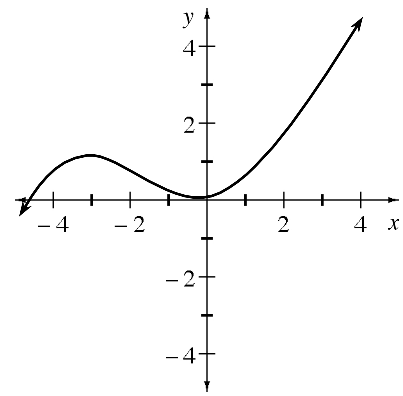


What happens when the change is made within the parentheses as in parts (c) and (d)? Here the shift is with the x-coordinates, thus the graph will move left or right.

In part (c), the graph of $f(x-1)$ is shifted to the **right** 1 unit.

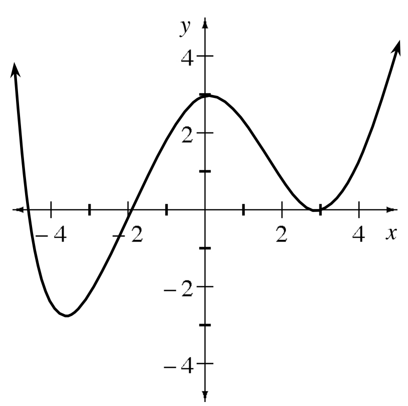


The graph in part (d), $f(x+3)$, is shifted to the **left** 3 units.

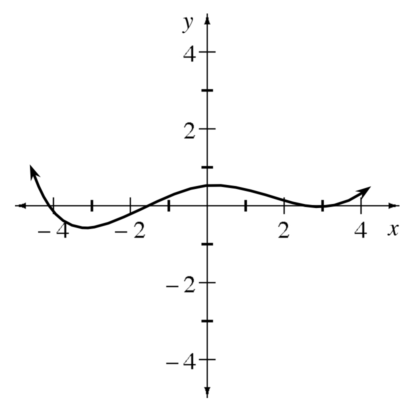


When multiplying $f(x)$ by a number as in parts (e) and (f), look at some key points. In particular, consider the x-intercepts. Since the y-value is zero at these points, multiplying by any number will not change the y-value. Therefore, the x-intercepts do not change at all, but the y-intercept will. In the original graph, the y-intercept is 1, so $f(0) = 1$. The larger the constant by which you multiply, the more stretched out the graph becomes. A smaller number flattens the graph.

Multiplying by 3 will raise that point three times as high, to the point (0, 3).



Multiplying by $\frac{1}{2}$ changes the y-intercept to $\left(0, \frac{1}{2}\right)$.



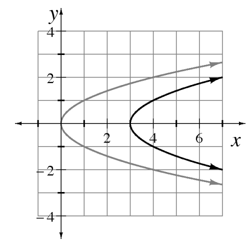
Example 3

Apply your knowledge of parent graphs and transformations to graph the following two non-functions.

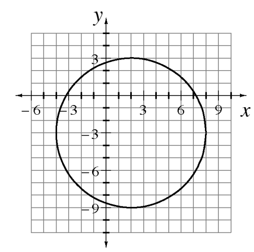
a. $x = y^2 + 3$ b. $(x-2)^2 + (y+3)^2 = 36$

Solution follows on next page →

Not every equation is a function, and the two non-functions students consider are $x = y^2$ and $x^2 + y^2 = r^2$. The first is the equation of a "sleeping parabola," or a parabola lying on its side. The second equation is the general form of a circle with center $(0, 0)$, and radius of length r. As written, neither of these equations can be entered into a graphing calculator. Students need to solve each of these as "$y =$" to use the calculator. But rather than doing that, the students can use what they have already learned to make accurate graphs of each equation. The parent of the equation in part (a) is $x = y^2$. The "+ 3" tells us the graph will shift 3 units, but is it up, down, left, or right? Rewriting the equation as $\pm\sqrt{x-3} = y$ helps us see that this graph is shifted to the right 3 units. At right, the grey curve is the graph of $x = y^2$, and the darker curve is the graph of $x = y^2 + 3$.



Graphing the equation of a circle is straightforward: a circle with center (h, k) and radius r has the equation $(x-h)^2 + (y+k)^2 = r^2$. Therefore the graph of the equation in part (b) is a circle with a center at $(2, -3)$, and a radius of 6. The graph of the circle is shown at right.



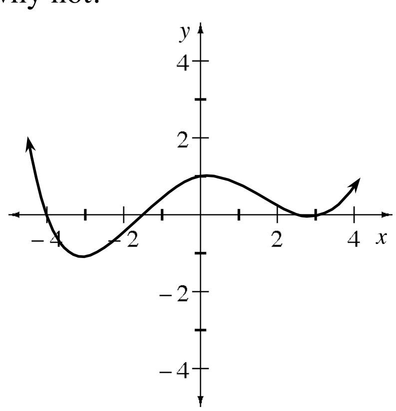
Problems

For each of the following equations, state the parent equation and then sketch its graph. Be sure to include any key and/or locator points.

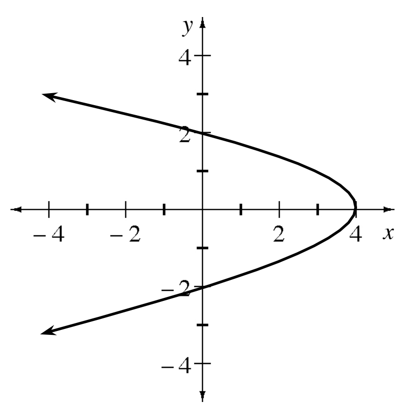
1. $y = (x-5)^2$
2. $y = -\frac{1}{3}(x+4)^2 + 7$
3. $(x-2)^2 + (y+1)^2 = 9$
4. $y = |x+5| - 2$
5. $y = \frac{1}{x+1} + 10$
6. $y = 2^x - 8$
7. $y = -(x-2)^3 + 1$
8. $y = \sqrt{x+7}$
9. $y = 3|x-5|$
10. $y = \pm\sqrt{x-9}$

For each of the following problems, state whether or not it is a function. If it is not a function, explain why not.

11.



12.

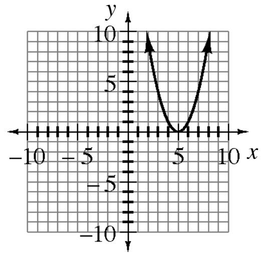


13. $y = 7 \pm \sqrt{9 - x^2}$

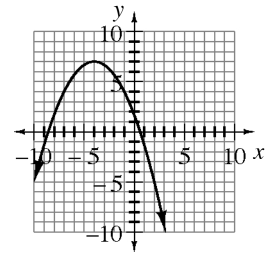
14. $y = 3(x-4)^2$

Answers

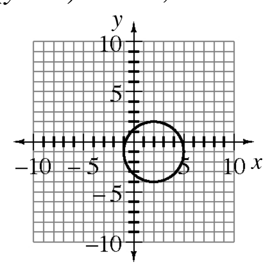
1. Parent graph $f(x)=x^2$, vertex $(5,0)$



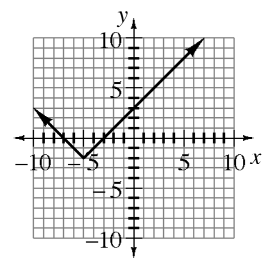
2. Parent graph $f(x)=x^2$, vertex $(-4,7)$



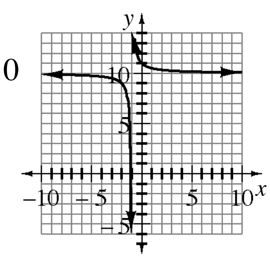
3. Parent graph $(x-h)^2+(y-k)^2=r^2$, center $(2,-1)$, radius 3



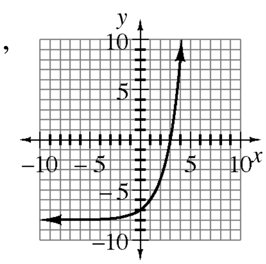
4. Parent graph $f(x)=|x|$, vertex $(-5,-2)$



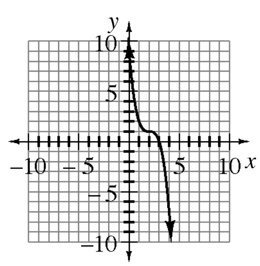
5. Parent graph $f(x)=\frac{1}{x}$, asymptotes $x=-1$, $y=10$



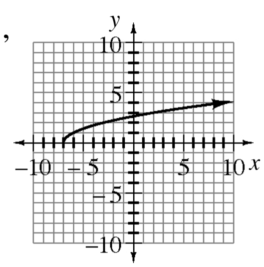
6. Parent graph $f(x)=2^x$, asymptote $x=-8$



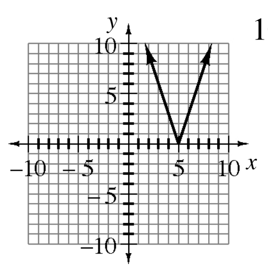
7. Parent graph $f(x)=x^3$, locator point $(2, 1)$



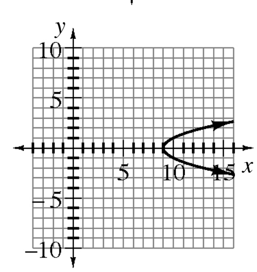
8. Parent graph $f(x)=\sqrt{x}$, vertex $(-7,0)$



9. Parent graph $f(x)=|x|$, vertex $(5,0)$



10. Parent graph $y^2=x$, vertex $(9,0)$



11. Yes.

12. No, on the left part of the graph, for each x-value there are two possible y-values. You can see this by drawing a vertical line through the graph. If a vertical line passes through the graph more than once, it is not a function.

13. No, because the equation has "±," for each value substituted for x, there will be two y-values produced. A function can have only one output for each input.

14. Yes.

MORE ON COMPLETING THE SQUARE

Although students can find the vertex of a parabola by averaging the x-intercepts, they also can use the algebraic method known as completing the square. This allows students to go directly from standard (or non-graphing) form to graphing form without the intermediate step of finding the x-intercepts. Completing the square is also used when the equation of a circle is written in an expanded form. When the students first looked at how to complete the square, they used tiles so that they could see how the method works. When they tried to create a square (complete it) by arranging the tiles, there were either too many or missing parts. This visual representation helps students see how to rewrite the equation algebraically.

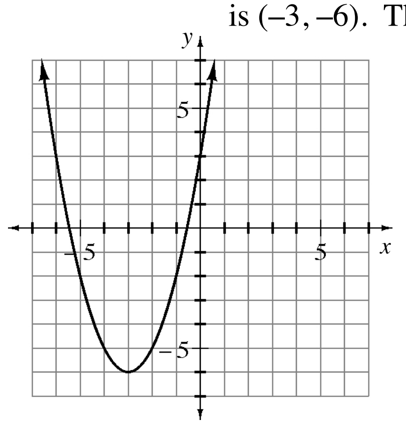
Example 1

The function $f(x) = x^2 + 6x + 3$ is written in standard form. Complete the square to write it in graphing form. Then state the vertex of the parabola and sketch the graph.

The general equation of a parabola in graphing form is $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex. The original equation needs to be changed into a set of parentheses squared, with a constant either added to or subtracted from it. To do this, we must know that $(x - h)^2 = x^2 - 2xh + h^2$. We will use this form of a perfect square to complete the square of the given equation or function.

$$\begin{aligned} f(x) &= x^2 + 6x + 3 \\ &= x^2 + 6x + \boxed{} + 3 - \boxed{} \\ &= x^2 + 6x + \boxed{9} + 3 - \boxed{9} \\ &= (x+3)^2 - 6 \end{aligned}$$

The first box holds a space for the number we have to add to complete the square. The second box is to subtract that same number so as not to change the balance of the equation. To determine the missing number, take half the coefficient of x (half of 6), and then square it and place the result in both boxes. With the equation in graphing form, we know the vertex is (–3, –6). The graph is shown below.



Example 2

The equation $x^2 - 8x + y^2 + 16y = 41$ is the equation of a circle. Complete the square to determine the coordinates of its center and the length of the radius.

As with the last example, we will fill in the blanks to create perfect squares. We need to do this twice: once for x, and again for y.

$$x^2 - 8x + y^2 + 16y = 41$$
$$x^2 - 8x + \boxed{} - \boxed{} + y^2 + 16y + \boxed{} - \boxed{} = 41$$
$$x^2 - 8x + \boxed{16} - \boxed{16} + y^2 + 16x + \boxed{64} - \boxed{64} = 41$$
$$(x-4)^2 - 16 + (y+8)^2 - 64 = 41$$
$$(x-4)^2 + (y+8)^2 = 41 + 16 + 64$$
$$(x-4)^2 + (y+8)^2 = 121$$

This is a circle with center (4, –8) and a radius of $\sqrt{121} = 11$.

Problems

Write each of the following equations in graphing form. Then state the vertex and the direction the parabola opens.

1. $y = x^2 - 8x + 18$
2. $y = -x^2 - 2x - 7$
3. $y = 3x^2 - 24x + 42$
4. $y = 2x^2 - 6$
5. $y = \frac{1}{2}x^2 - 3x + \frac{1}{2}$
6. $y = x^2 + 18x + 97$

Find the center and radius of each circle.

7. $(x+2)^2 + (y+7)^2 = 25$
8. $3(x-9)^2 + 3(y+1)^2 = 12$
9. $x^2 + 6x + y^2 = 91$
10. $x^2 - 10x + y^2 + 14y = -58$
11. $x^2 + 50x + y^2 - 2y = -602$
12. $x^2 + y^2 - 8x - 16y = 496$

Answers

1. $y = (x-4)^2 + 2$, vertex (4, 2), up
2. $y = -(x+1)^2 - 6$, vertex (–1, –6), down
3. $y = 3(x-4)^2 - 6$, vertex (4, –6), up
4. $y = 2(x-0)^2 - 6$, vertex (0, –6), up
5. $y = \frac{1}{2}(x-3)^2 - 4$, vertex (3, –4), up
6. $y = (x+9)^2 + 16$, vertex (–9, –16), up
7. Center: (–2, –7), radius: 5
8. Center: (9, –1), radius: 2
9. Center: (–3, 0), radius: 10
10. Center: (5, –7), radius: 4
11. Center: (–25, 1), radius: $\sqrt{24} = 2\sqrt{6}$
12. Center: (4, 8), radius: 24

SAT PREP

1. If m is an integer, which of the following could **not** equal m^3?

   a. 27 b. 0 c. 1 d. 16 e. 64

2. If n is divided by 7 the remainder is 3. What is the remainder if $3n$ is divided by 7?

   a. 2 b. 3 c. 4 d. 5 e. 6

3. What is the slope of the line passing through the point $(-3, -1)$ and the origin?

   a. -3 b. $-\frac{1}{3}$ c. 0 d. $\frac{1}{3}$ e. 3

4. If $x = 2y + 3$ and $3x = 7 - 4y$, what does x equal?

   a. -5 b. $-\frac{1}{5}$ c. $\frac{13}{5}$ d. $\frac{2}{3}$ e. 15

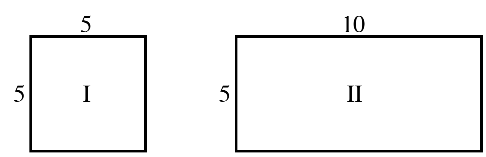
5. A bag contains a number of marbles of which 35 are blue, 16 are red and the rest are yellow. If the probability of selecting a yellow marble from the bag at random is $\frac{1}{4}$, how many yellow marbles are in the bag?

   a. 4 b. 17 c. 19 d. 41 e. 204

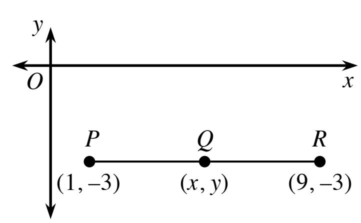
6. If $n > 0$ and $16x^2 + kx + 25 = (4x + n)^2$ for all values of x, what does $k - n$ equal?

   a. 0 b. 5 c. 35 d. 40 e. 80

7. A rectangular solid has two faces congruent to figure I at right and four faces congruent to figure II at right. What is the volume of the solid?

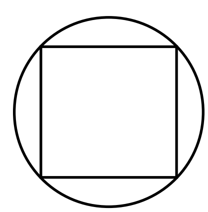


8. In the figure at right, $PQ = QR$. What is the x-coordinate of point Q?



9. The time t, in hours, needed to produce u units of a product is given by the formula $t = ku + c$, where k and c are constants. If it takes 430 hours to produce 100 units and 840 hours to produce 200 units, what is the value of c?

10. In the figure at right, a square is inscribed in a circle. If the sides of the square measure $\sqrt{3}$ and the area of the circle is $c\pi$, what is the exact value of c?



Answers

1. D
2. A
3. D
4. C
5. B
6. C
7. 250
8. 5
9. 20
10. 1.5

SAT PREP

1. If m is an integer, which of the following could **not** equal m^3?

   a. 27 b. 0 c. 1 d. 16 e. 64

2. If n is divided by 7 the remainder is 3. What is the remainder if $3n$ is divided by 7?

   a. 2 b. 3 c. 4 d. 5 e. 6

3. What is the slope of the line passing through the point $(-3, -1)$ and the origin?

   a. -3 b. $-\frac{1}{3}$ c. 0 d. $\frac{1}{3}$ e. 3

4. If $x = 2y + 3$ and $3x = 7 - 4y$, what does x equal?

   a. -5 b. $-\frac{1}{5}$ c. $\frac{13}{5}$ d. $\frac{2}{3}$ e. 15

5. A bag contains a number of marbles of which 35 are blue, 16 are red and the rest are yellow. If the probability of selecting a yellow marble from the bag at random is $\frac{1}{4}$, how many yellow marbles are in the bag?

   a. 4 b. 17 c. 19 d. 41 e. 204

6. If $n > 0$ and $16x^2 + kx + 25 = (4x + n)^2$ for all values of x, what does $k - n$ equal?

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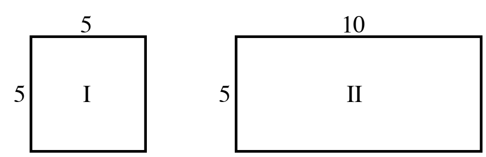

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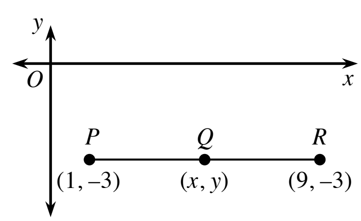

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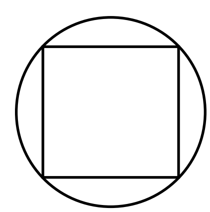


Answers

1. D
2. A
3. D
4. C
5. B
6. C
7. 250
8. 5
9. 20
10. 1.5

EQUIVALENCE 3.1.1 – 3.1.3

Solving equations is a skill that algebra students practice a great deal. In Algebra 2, the equations become increasingly more complex. Whenever possible, it is beneficial for students to rewrite equations in a simpler form, or as equations they already know how to solve. This is done by recognizing equivalent expressions and developing algebraic strategies for demonstrating equivalence.

Example 1

Emma and Rueben have been given a sequence they have never seen before. It does not seem to be an arithmetic or a geometric sequence.

n	$t(n)$
1	–36
2	–40
3	–42
4	–42
5	–40
6	–36

After a great deal of brain strain, Emma exclaims, *"Hey! I see a pattern! If we look at the differences between the t(n)'s, we can list the whole sequence!"* Rueben agrees, *"Oh, I see it! But what a pain to list it all out. We should be able to find a formula."*

"Now that we see a pattern," Emma says, *"Let's each spend some time thinking of a formula."*

After a few minutes, both Rueben and Emma have formulas. *"Wait a minute!"* says Rueben. *"That's not the formula I got. My formula is* $t(n) = n^2 - 7n - 30$*, but your formula is* $t(n) = (n + 3)(n - 10)$. *Which one of us is correct?"*

a. Who is correct? Justify your answer completely.

b. Later Tess, another team member, says, *"Ha! I have the right equation! It is* $t(n) = \left(n - \frac{7}{2}\right)^2 - \frac{169}{4}$.*"* Rueben comments, *"You are really off, Tess. That is nowhere near the right answer!"* Is Rueben correct, or has Tess found another equation? Justify your answer.

We can show that both Rueben and Emma's equations produce the values in the table by substituting different values for n, but that would only show that they are equivalent for those specific values. We must show that the two equations are equivalent algebraically in order to verify that they are the same. To do this, we can use algebra to rewrite one equation, and hopefully get the other one.

$$\begin{aligned} t(n) &= (n+3)(n-10) \\ &= n^2 - 10n + 3n - 30 \\ &= n^2 - 7n - 30 \end{aligned}$$

We started with Rueben's equation, and through algebraic manipulation, the result was Emma's equation. Therefore the equations are equivalent.

Similarly, we can manipulate Tess' equation to see if we can get either of the other two equations. If we can, then it too is equivalent. Start by expanding $\left(n-\frac{7}{2}\right)^2$.

$$\begin{aligned} t(n) &= \left(n-\tfrac{7}{2}\right)^2 - \tfrac{169}{4} \\ &= n^2 - 2\left(\tfrac{7}{2}\right)(n) + \tfrac{49}{4} - \tfrac{169}{4} \\ &= n^2 - 7n - \tfrac{120}{4} \\ &= n^2 - 7n - 30 \end{aligned}$$

Tess has an equivalent equation as well. Therefore all three are equivalent equations.

Example 2

Solve the following equation by rewriting it as a simpler equivalent equation: $\frac{1}{32}x^2 - \frac{1}{8}x = 1$.

This equation would be much simpler if it did not have any fractions, so multiply everything by 32 to eliminate the denominators.

$$\begin{aligned} \tfrac{1}{32}x^2 - \tfrac{1}{8}x &= 1 \\ 32\left(\tfrac{1}{32}x^2 - \tfrac{1}{8}x\right) &= 32(1) \\ 32\left(\tfrac{1}{32}x^2\right) - 32\left(\tfrac{1}{8}x\right) &= 32 \\ x^2 - 4x &= 32 \end{aligned}$$

To solve a quadratic equation, set it equal to zero, and solve by either factoring or using the Quadratic Formula. Since the equation appears to be easily factorable, use that method.

$$\begin{aligned} x^2 - 4x &= 32 \\ x^2 - 4x - 32 &= 0 \\ (x-8)(x+4) &= 0 \\ x-8=0,\ x+4 &= 0 \\ x=8,\ x &= -4 \end{aligned}$$

Example 3

Decide whether or not the following pairs of expressions or equations are equivalent for all values of the variables. Justify your answer completely.

a. $\sqrt{a+b}$ and $\sqrt{a}+\sqrt{b}$

b. $\frac{12}{x+4}$ and $\frac{12}{x}+\frac{12}{4}$

c. $\frac{x+4}{12}$ and $\frac{x}{12}+\frac{4}{12}$

d. $3x^2 + 6x - 1 = x^2 + 18x - 14$ and $2x^2 - 12x + 13 = 0$

In part (a), choose different values for a and b to check. For instance, if $a = 1$ and $b = 2$, then we would have $\sqrt{1+2}=\sqrt{3}\approx 1.732$, and $\sqrt{1}+\sqrt{2}=1+\sqrt{2}\approx 2.414$. Therefore the two expressions are not equal. (Note: These expressions are only equal when both a and b are equal to zero.)

Try any value for x in part (b), and the two expressions will not be equal. For example, if $x = 1$, then $\frac{12}{x+4}=\frac{12}{1+4}=\frac{12}{5}$ and $\frac{12}{x}+\frac{12}{4}=\frac{12}{1}+\frac{12}{4}=12+3=15$.

Note that you only need to find *one* example the does not work to demonstrate that the two expressions of equations are not equivalent. This strategy is known as a counterexample.

The expressions in part (c) demonstrate how we add fractions with common denominators: by adding the numerators.

You may wish to try some values of x in the two equations of part (d), but the equations are fairly messy. In addition, using a few values would not show that the equations are equivalent for *all* values of x. It is easier to simplify the first equation to see if it results in the second equation.

$$3x^2+6x-1=x^2+18x-14$$
$$2x^2+6x-1=18x-14$$
$$2x^2-12x-1=-14$$
$$2x^2-12x+13=0$$

The result is the second equation. Therefore, these two equations are equivalent.

Problems

Rewrite the following expressions in a simpler form.

1. $(3x^{-2})^{-5}(4x^{3})$

2. $\dfrac{3x^{-4}y^{3}}{x^{3}y^{-5}}$

3. $\left[\dfrac{-27x^{-12}y^{8}z}{\left(-3x^{4}y^{2}z\right)^{-2}}\right]^{0}$

Decide whether or not the following pairs of expressions or equations are equivalent for any values of the variables. Justify your answer completely.

4. $3x + 3 = 6x + 6$ and $x + 3 = x + 6$

5. $3x + 4y = 12$ and $y = \frac{3}{4}x - 3$

6. $(0.5x + 1)(0.5x - 2) = 0$ and $2x^{2} - 4x - 16 = 0$

7. $y - 9 = -\frac{3}{2}(x - 2)$ and $y - 3 = -\frac{3}{2}(x - 8)$

For each sequence below there are two equations. Decide whether or not the equations represent the sequence, and whether or not the equations are equivalent. Justify your answer.

8.

n	0	1	2	3	4
$t(n)$	14	11	8	5	2

A: $t(n) = -3n + 14$

B: $t(n) = -3(n - 4) + 2$

9.

n	1	2	3	4	5
$t(n)$	128	64	32	16	8

A: $t(n) = 2^{8-n}$

B: $t(n) = 128\left(\frac{1}{2}\right)^{n-1}$

10.

n	1	3	5	7	9
$t(n)$	9	49	121	225	361

A: $t(n) = (2n + 1)^{2}$

B: $t(n) = 4\left(n + \frac{1}{2}\right)^{2}$

11.

n	0	1	2	3	4
$t(n)$	8	$\frac{8}{3}$	$\frac{8}{9}$	$\frac{8}{27}$	$\frac{8}{81}$

A: $t(n) = \left(\frac{8}{3}\right)^{n}$

B: $t(n) = 8\left(\frac{1}{3}\right)^{n}$

Simplify, and then solve the following equations.

12. $100x^{2} + 500x = -600$

13. $4x + 2y = 30$
    $2x - y = 5$

14. $\dfrac{x-1}{11} - \dfrac{7}{66} = \dfrac{1}{6}$

15. $\dfrac{x^{2}+3x+2}{x^{2}-x-6} + \dfrac{x^{2}+4x-5}{x^{2}+2x-15} = \dfrac{x^{2}+6x}{x^{2}+4x-12}$

Answers

1. $\frac{4x^{13}}{243}$

2. $\frac{3y^8}{x^7}$

3. 1

4. No, in the first equation, $x = -1$. The second equation has no solution.

5. No, (0, 3) works in the first equation but not in the second. The standard form of the second equation is $3x - 4y = 12$.

6. Equivalent. Multiply the two binomials in the first equation go get $0.25x^2 - 0.5x - 2 = 0$. Then multiply all of the terms by 8.

7. No, rewriting the equations in slope-intercept form produces different y-intercepts.

8. A and B both represent the sequence, and are equivalent.

9. A and B both represent the sequence, and are equivalent.

10. A and B both represent the sequence, and are equivalent.

11. Only B represents the sequence. They are not equivalent.

12. $x^2 + 5x + 6 = 0$, $x = -2$, $x = -3$

13. (5, 5)

14. $x = 4$

15. $x = 0, 1$

SIMPLIFYING EXPRESSIONS 3.2.2

To **simplify rational expressions**, find factors in the numerator and denominator that are the same and then write them as fractions equal to 1. For example,

$$\frac{6}{6}=1 \qquad \frac{x^2}{x^2}=1 \qquad \frac{(x+2)}{(x+2)}=1 \qquad \frac{(3x-2)}{(3x-2)}=1$$

Notice that the last two examples involved binomial sums and differences. **Only** when sums or differences are **exactly** the same does the fraction equal 1. The rational expressions below **cannot** be simplified:

$$\frac{(6+5)}{6} \qquad \frac{x^3+y}{x^3} \qquad \frac{x}{x+2} \qquad \frac{3x-2}{2}$$

As shown in the examples below, most problems that involve simplifying rational expressions will require that you **factor** the numerator and denominator.

Note that in all cases we assume the denominator does not equal zero, so in Example 4 below the simplification is only valid provided $x \neq -6$ or 2. For more information, see examples 1 and 2 in the Math Notes box in Lesson 3.2.4.

One other special situation is shown in the following examples:

$$\frac{-2}{2}=-1 \qquad \frac{-x}{x}=-1 \qquad \frac{-x-2}{x+2} \Rightarrow \frac{-(x+2)}{x+2} \Rightarrow -1 \qquad \frac{5-x}{x-5} \Rightarrow \frac{-(x-5)}{x-5} \Rightarrow -1$$

Again assume the denominator does not equal zero.

Example 1

$$\frac{12}{54}=\frac{2\cdot 2\cdot 3}{2\cdot 3\cdot 3\cdot 3}=\frac{2}{9} \text{ since } \frac{2}{2}=\frac{3}{3}=1$$

Example 2

$$\frac{6x^3y^2}{15x^2y^4}=\frac{2\cdot 3\cdot x^2\cdot x\cdot y^2}{5\cdot 3\cdot x^2\cdot y^2\cdot y^2}=\frac{2x}{5y^2}$$

Example 3

$$\frac{12(x-1)^3(x+2)}{3(x-1)^2(x+2)^2}=\frac{4\cdot 3(x-1)^2(x-1)(x+2)}{3(x-1)^2(x+2)(x+2)}$$

$$=\frac{4(x-1)}{(x+2)} \text{ since } \frac{3}{3}, \frac{(x-1)^2}{(x-1)^2}, \text{ and } \frac{x+2}{x+2}=1$$

Example 4

$$\frac{x^2-6x+8}{x^2+4x-12}=\frac{(x-4)(x-2)}{(x+6)(x-2)}$$

$$=\frac{(x-4)}{(x+6)} \text{ since } \frac{(x-2)}{(x-2)}=1$$

Problems

Simplify each of the following expressions completely. Assume the denominator does not equal zero.

1. $\frac{2(x+3)}{4(x-2)}$
2. $\frac{2(x-3)}{6(x+2)}$
3. $\frac{2(x+3)(x-2)}{6(x-2)(x+2)}$
4. $\frac{4(x-3)(x-5)}{6(x-3)(x+2)}$
5. $\frac{3(x-3)(4-x)}{15(x+3)(x-4)}$
6. $\frac{15(x-1)(7-x)}{25(x+1)(x-7)}$
7. $\frac{24(y-4)(y-6)}{16(y+6)(6-y)}$
8. $\frac{36(y+4)(y-16)}{32(y+16)(16-y)}$
9. $\frac{(x+3)^2(x-2)^4}{(x+3)^4(x-2)^3}$
10. $\frac{(5-x)^2(x-2)^2}{(x+5)^4(x-2)^3}$
11. $\frac{(5-x)^4(3x-1)^2}{(x-5)^4(3x-2)^3}$
12. $\frac{12(x-7)(x+2)^4}{20(x-7)^2(x+2)^5}$
13. $\frac{x^2+5x+6}{x^2+x-6}$
14. $\frac{2x^2+x-3}{x^2+4x-5}$
15. $\frac{x^2+4x}{2x+8}$
16. $\frac{24(3x-7)(x+1)^6}{20(3x-7)^3(x+1)^5}$
17. $\frac{x^2-1}{(x+1)(x-2)}$
18. $\frac{x^2-4}{(x+1)^2(x-2)}$
19. $\frac{x^2-4}{x^2+x-6}$
20. $\frac{x^2-16}{x^3+9x^2+20x}$
21. $\frac{2x^2-x-10}{3x^2+7x+2}$

Answers

1. $\frac{(x+3)}{2(x-2)}$
2. $\frac{(x-3)}{3(x+2)}$
3. $\frac{(x+3)}{3(x+2)}$
4. $\frac{2(x-5)}{3(x+2)}$
5. $-\frac{(x-3)}{5(x+3)}$
6. $-\frac{3(x-1)}{5(x+1)}$
7. $-\frac{3(y-4)}{2(y+6)}$
8. $-\frac{9(y+4)}{8(y+16)}$
9. $\frac{(x-2)}{(x+3)^2}$
10. $\frac{(5-x)^2}{(x+5)^4(x-2)}$
11. $\frac{(3x-1)^2}{(3x-2)^3}$
12. $\frac{3}{5(x-7)(x+2)}$
13. $\frac{x+2}{x-2}$
14. $\frac{2x+3}{x+5}$
15. $\frac{x}{2}$
16. $\frac{6(x+1)}{5(3x-7)^2}$
17. $\frac{x-1}{x-2}$
18. $\frac{x+2}{(x+1)^2}$
19. $\frac{x+2}{x+3}$
20. $\frac{x-4}{x(x+5)}$
21. $\frac{2x-5}{3x+1}$

MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS 3.2.3

Multiplication or division of rational expressions follows the same procedure used with numerical fractions. However, it is often necessary to factor the polynomials in order to simplify them. When dividing rational expressions, change the problem to multiplication by inverting (flipping) the second expression (or any expression that follows a division sign) and completing the process as you do for multiplication. As in the previous section, remember that simplification assumes that the denominator is not equal to zero. For addition information, see Examples 3 and 4 in the Math Notes box in Lesson 3.2.4.

Example 1

Multiply $\frac{x^2+6x}{(x+6)^2} \cdot \frac{x^2+7x+6}{x^2-1}$ and simplify the result.

After factoring, the expression becomes: $\frac{x(x+6)}{(x+6)(x+6)} \cdot \frac{(x+6)(x+1)}{(x+1)(x-1)}$

After multiplying, reorder the factors: $\frac{(x+6)}{(x+6)} \cdot \frac{(x+6)}{(x+6)} \cdot \frac{x}{(x-1)} \cdot \frac{(x+1)}{(x+1)}$

Since $\frac{(x+6)}{(x+6)} = 1$ and $\frac{(x+1)}{(x+1)} = 1$, simplify: $1 \cdot 1 \cdot \frac{x}{x-1} \cdot 1 \Rightarrow \frac{x}{x-1}$ for $x \neq 6, -1$, or 1

Example 2

Divide $\frac{x^2-4x-5}{x^2-4x+4} \div \frac{x^2-2x-15}{x^2+4x-12}$ and simplify the result.

First change to a multiplication expression by inverting (flipping) the second fraction: $\frac{x^2-4x-5}{x^2-4x+4} \cdot \frac{x^2+4x-12}{x^2-2x-15}$

After factoring, the expression is: $\frac{(x-5)(x+1)}{(x-2)(x-2)} \cdot \frac{(x+6)(x-2)}{(x-5)(x+3)}$

Reorder the factors (if you need to): $\frac{(x-5)}{(x-5)} \cdot \frac{(x-2)}{(x-2)} \cdot \frac{(x+1)}{(x-2)} \cdot \frac{(x+6)}{(x+3)}$

Since $\frac{(x-5)}{(x-5)} = 1$ and $\frac{(x-2)}{(x-2)} = 1$, simplify: $\frac{(x+1)}{(x-2)} \cdot \frac{(x+6)}{(x+3)}$

Thus, $\frac{x^2-4x-5}{x^2-4x+4} \div \frac{x^2-2x-15}{x^2+4x-12} = \frac{(x+1)}{(x-2)} \cdot \frac{(x+6)}{(x+3)}$ or $\frac{x^2+7x+6}{x^2+x-6}$ for $x \neq -3, 2$, or 5.

Problems

Multiply or divide each pair of rational expressions. Simplify the result. Assume the denominator is not equal to zero.

1. $\frac{x^2+5x+6}{x^2-4x}\cdot\frac{4x}{x+2}$

2. $\frac{x^2-2x}{x^2-4x+4}\div\frac{4x^2}{x-2}$

3. $\frac{x^2-16}{(x-4)^2}\cdot\frac{x^2-3x-18}{x^2-2x-24}$

4. $\frac{x^2-x-6}{x^2+3x-10}\cdot\frac{x^2+2x-15}{x^2-6x+9}$

5. $\frac{x^2-x-6}{x^2-x-20}\cdot\frac{x^2+6x+8}{x^2-x-6}$

6. $\frac{x^2-x-30}{x^2+13x+40}\cdot\frac{x^2+11x+24}{x^2-9x+18}$

7. $\frac{15-5x}{x^2-x-6}\div\frac{5x}{x^2+6x+8}$

8. $\frac{17x+119}{x^2+5x-14}\div\frac{9x-1}{x^2-3x+2}$

9. $\frac{2x^2-5x-3}{3x^2-10x+3}\cdot\frac{9x^2-1}{4x^2+4x+1}$

10. $\frac{x^2-1}{x^2-6x-7}\div\frac{x^3+x^2-2x}{x-7}$

11. $\frac{3x-21}{x^2-49}\div\frac{3x}{x^2+7x}$

12. $\frac{x^2-y^2}{x+y}\cdot\frac{1}{x-y}$

13. $\frac{y^2-y}{w^2-y^2}\div\frac{y^2-2y+1}{1-y}$

14. $\frac{y^2-y-12}{y+2}\div\frac{y-4}{y^2-4y-12}$

15. $\frac{x^2+7x+10}{x+2}\div\frac{x^2+2x-15}{x+2}$

Answers

1. $\frac{4(x+3)}{x-4}$

2. $\frac{1}{4x}$

3. $\frac{x+3}{x-4}$

4. $\frac{x+2}{x-2}$

5. $\frac{x+2}{x-5}$

6. $\frac{x+3}{x-3}$

7. $\frac{-x-4}{x}$

8. $\frac{17(x-1)}{9x-1}$

9. $\frac{3x+1}{2x+1}$

10. $\frac{1}{x(x+2)}$

11. 1

12. 1

13. $\frac{-y}{w^2-y^2}$

14. $(y+3)(y-6)$

15. $\frac{x+2}{x-3}$

ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS 3.2.4

Addition and Subtraction of Rational Expressions uses the same process as simple numerical fractions. First, find a common denominator (if necessary). Second, convert the original fractions to equivalent ones with the common denominator. Third, add (or subtract) the new numerators over the common denominator. Finally, factor the numerator and denominator and reduce (if possible). For additional information, see the Math Notes box in Lesson 3.2.5. Note that these steps are only valid provided that the denominator is not zero.

Example 1

The least common multiple of $2(n + 2)$ and $n(n + 2)$ is $2n(n + 2)$.

$$\frac{3}{2(n+2)}+\frac{3}{n(n+2)}$$

To get a common denominator in the first fraction, multiply the fraction by $\frac{n}{n}$, a form of the number 1. Multiply the second fraction by $\frac{2}{2}$.

$$=\frac{3}{2(n+2)}\cdot\frac{n}{n}+\frac{3}{n(n+2)}\cdot\frac{2}{2}$$

Multiply the numerator and denominator of each term. It may be necessary to distribute the numerator.

$$=\frac{3n}{2n(n+2)}+\frac{6}{2n(n+2)}$$

Add, factor, and simplify the result. (Note: $n \neq 0$ or 2)

$$=\frac{3n+6}{2n(n+2)}\Rightarrow\frac{3(n+2)}{2n(n+2)}\Rightarrow\frac{3}{2n}$$

Example 2

$$\frac{2-x}{x+4}+\frac{3x+6}{x+4}\quad\Rightarrow\quad\frac{2-x+3x+6}{x+4}\quad\Rightarrow\quad\frac{2x+8}{x+4}\quad\Rightarrow\quad\frac{2(x+4)}{x+4}\quad\Rightarrow\quad 2$$

Example 3

$$\frac{3}{x-1}-\frac{2}{x-2}\quad\Rightarrow\quad\frac{3}{x-1}\cdot\frac{x-2}{x-2}-\frac{2}{x-2}\cdot\frac{x-1}{x-1}\quad\Rightarrow\quad\frac{3x-6-2x+2}{(x-1)(x-2)}\quad\Rightarrow\quad\frac{x-4}{(x-1)(x-2)}$$

Problems

Add or subtract each expression and simplify the result. In each case assume the denominator does not equal zero.

1. $\dfrac{x}{(x+2)(x+3)}+\dfrac{2}{(x+2)(x+3)}$
2. $\dfrac{x}{x^2+6x+8}+\dfrac{4}{x^2+6x+8}$
3. $\dfrac{b^2}{b^2+2b-3}+\dfrac{-9}{b^2+2b-3}$
4. $\dfrac{2a}{a^2+2a+1}+\dfrac{2}{a^2+2a+1}$
5. $\dfrac{x+10}{x+2}+\dfrac{x-6}{x+2}$
6. $\dfrac{a+2b}{a+b}+\dfrac{2a+b}{a+b}$
7. $\dfrac{3x-4}{3x+3}-\dfrac{2x-5}{3x+3}$
8. $\dfrac{3x}{4x-12}-\dfrac{9}{4x-12}$
9. $\dfrac{6a}{5a^2+a}-\dfrac{a-1}{5a^2+a}$
10. $\dfrac{x^2+3x-5}{10}-\dfrac{x^2-2x+10}{10}$
11. $\dfrac{6}{x(x+3)}+\dfrac{2x}{x(x+3)}$
12. $\dfrac{5}{x-7}+\dfrac{3}{4(x-7)}$
13. $\dfrac{5x+6}{x^2}-\dfrac{5}{x}$
14. $\dfrac{2}{x+4}-\dfrac{x-4}{x^2-16}$
15. $\dfrac{10a}{a^2+6a}-\dfrac{3}{3a+18}$
16. $\dfrac{3x}{2x^2-8x}+\dfrac{2}{x-4}$
17. $\dfrac{5x+9}{x^2-2x-3}+\dfrac{6}{x^2-7x+12}$
18 $\dfrac{x+4}{x^2-3x-28}-\dfrac{x-5}{x^2+2x-35}$
19. $\dfrac{3x+1}{x^2-16}-\dfrac{3x+5}{x^2+8x+16}$
20. $\dfrac{7x-1}{x^2-2x-3}-\dfrac{6x}{x^2-x-2}$

Answers

1. $\dfrac{1}{x+3}$
2. $\dfrac{1}{x+2}$
3. $\dfrac{b-3}{b-1}$
4. $\dfrac{2}{a+1}$
5. 2
6. 3
7. $\dfrac{1}{3}$
8. $\dfrac{3}{4}$
9. $\dfrac{1}{a}$
10. $\dfrac{x-3}{2}$
11. $\dfrac{2}{x}$
12. $\dfrac{23}{4(x-7)}=\dfrac{23}{4x-28}$
13. $\dfrac{6}{x^2}$
14. $\dfrac{1}{x+4}$
15. $\dfrac{9}{a+6}$
16. $\dfrac{7}{2(x-4)}=\dfrac{7}{2x-8}$
17. $\dfrac{5(x+2)}{(x-4)(x+1)}=\dfrac{5x+10}{x^2-3x-4}$
18. $\dfrac{14}{(x-7)(x+7)}=\dfrac{14}{x^2-49}$
19. $\dfrac{4(5x+6)}{(x-4)(x+4)^2}$
20. $\dfrac{x+2}{(x-3)(x-2)}=\dfrac{x+2}{x^2-5x+6}$

SAT PREP

1. If $-1 < t < 0$, which of the following statements must be true?

   a. $t^3 < t < t^2$ b. $t^2 < t^3 < t$ c. $t^2 < t < t^3$

   d. $t < t^3 < t^2$ e. $t < t^2 < t^3$

2. Without taking a single break, Mercedes hiked for 10 hours, up a mountain and back down by the same path. While hiking, she averaged 2 miles per hour on the way up and 3 miles per hour on the way down. How many miles was it from the base of the mountain to the top?

   a. 4   b. 6   c. 9   d. 12   e. 18

3. When a certain rectangle is folded in half, it forms two squares. If the perimeter of one of these squares is 28, what is the perimeter of the original rectangle?

   a. 30   b. 42   c. 49   d. 56

   e. Cannot be determined from the information given.

4. A class of 50 girls and 60 boys sponsored a road rally race. If 60% of the girls and 50% of the boys participated in the road rally, what percent of the class participated in the road rally?

   a. 54.5%   b. 55%   c. 57.5%   d. 88%   e. 110%

5. The sum of four consecutive integers is s. In terms of s, which of the following is the smallest of these four integers?

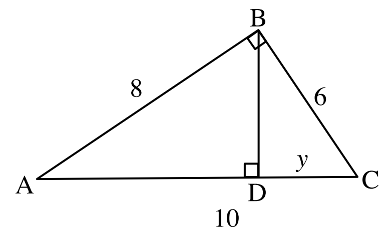
   a. $\frac{s-6}{4}$ b. $\frac{s-4}{4}$ c. $\frac{s-3}{4}$ d. $\frac{s-2}{4}$ e. $\frac{s}{4}$

6. On a certain map, 30 miles is represented by one-half inch. On the same map, how many miles are represented by 2.5 inches?

7. How many of the first one hundred positive integers contain the digit 9?

8. The sum of n and $n + 1$ is greater than five but less than 15. If n is an integer, what is one possible value of n?

9. In the figure at right, ΔABC is a right triangle and $\frac{y}{6} = \frac{6}{10}$. What is the value of y?



10. For three numbers a, b, and c, the average (arithmetic mean) is twice the median. If $a < b < c$, $a = 0$, and $c = kb$, what is the value of k?

Answers

1. D
2. D
3. B
4. A
5. A
6. 150 miles
7. 19 integers
8. n can be 3, 4, 5, or 6
9. $y = 3.6$
10. $k = 5$

SAT PREP

1. If $-1 < t < 0$, which of the following statements must be true?

   a. $t^3 < t < t^2$ b. $t^2 < t^3 < t$ c. $t^2 < t < t^3$

   d. $t < t^3 < t^2$ e. $t < t^2 < t^3$

2. Without taking a single break, Mercedes hiked for 10 hours, up a mountain and back down by the same path. While hiking, she averaged 2 miles per hour on the way up and 3 miles per hour on the way down. How many miles was it from the base of the mountain to the top?

   a. 4   b. 6   c. 9   d. 12   e. 18

3. When a certain rectangle is folded in half, it forms two squares. If the perimeter of one of these squares is 28, what is the perimeter of the original rectangle?

   a. 30   b. 42   c. 49   d. 56

   e. Cannot be determined from the information given.

4. A class of 50 girls and 60 boys sponsored a road rally race. If 60% of the girls and 50% of the boys participated in the road rally, what percent of the class participated in the road rally?

   a. 54.5%   b. 55%   c. 57.5%   d. 88%   e. 110%

5. The sum of four consecutive integers is s. In terms of s, which of the following is the smallest of these four integers?

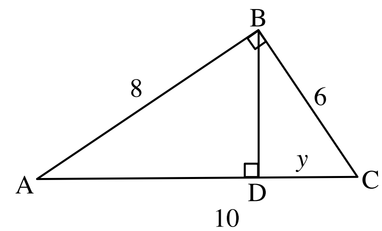
   a. $\frac{s-6}{4}$ b. $\frac{s-4}{4}$ c. $\frac{s-3}{4}$ d. $\frac{s-2}{4}$ e. $\frac{s}{4}$

6. On a certain map, 30 miles is represented by one-half inch. On the same map, how many miles are represented by 2.5 inches?

7. How many of the first one hundred positive integers contain the digit 9?

8. The sum of n and $n + 1$ is greater than five but less than 15. If n is an integer, what is one possible value of n?

9. In the figure at right, ΔABC is a right triangle and $\frac{y}{6} = \frac{6}{10}$. What is the value of y?



10. For three numbers a, b, and c, the average (arithmetic mean) is twice the median. If $a < b < c$, $a = 0$, and $c = kb$, what is the value of k?

Answers

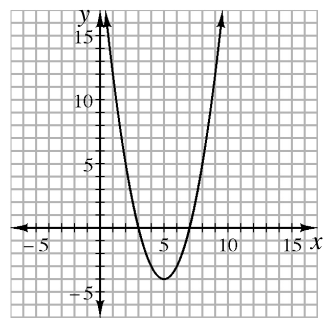
1. D
2. D
3. B
4. A
5. A
6. 150 miles
7. 19 integers
8. n can be 3, 4, 5, or 6
9. $y = 3.6$
10. $k = 5$

SOLVING SYSTEMS OF EQUATIONS 4.1.1 – 4.1.4

Students have been solving equations even before Algebra 1. Now they focus on what a solution means, both algebraically and graphically. By understanding the nature of solutions, students are able to solve equations in new and different ways. Their understanding also provides opportunities to solve some challenging applications. In this section they will extend their knowledge about solving one and two variable equations to solve systems with three variables.

Example 1

The graph of $y = (x - 5)^2 - 4$ is shown below right. Solve each of the following equations. Explain how the graph can be used to solve the equations.



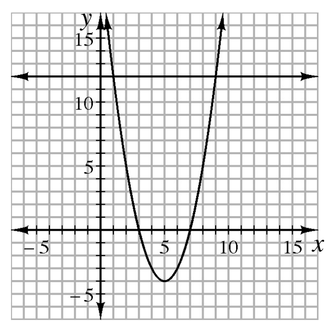
a. $(x - 5)^2 - 4 = 12$

b. $(x - 5)^2 - 4 = -3$

c. $(x - 5)^2 = 4$

$$\begin{aligned}(x-5)^2 - 4 &= 12\\ (x-5)^2 &= 16\\ x-5 &= \pm 4\\ x &= 5 \pm 4\\ x &= 9,\ 1\end{aligned}$$

Students can determine the correct answers through a variety of different ways. For part (a), most students would do the following:



This is correct and a standard procedure. However, with the graph of the parabola provided, the student can find the solution by inspecting the graph. Since we already have the graph of $y = (x - 5)^2 - 4$, we can add the graph of $y = 12$ which is a horizontal line. These two graphs cross at two points, and the x-coordinates of these points are the solutions. The intersection points are (1, 12) and (9, 12). Therefore the solutions to the equation are $x = 1$ and $x = 9$.

We can use this method for part (b) as well. Draw the graph of $y = -3$ to find that the graphs intersect at (4, –3) and (6, –3). Therefore the solutions to part (b) are $x = 4$ and $x = 6$.

The equation in part (c) might look as if we cannot solve it with the graph, but we can. Granted, this is easy to solve algebraically:

$$\begin{aligned}(x-5)^2 &= 4\\ x-5 &= \pm 2\\ x &= 7,\ 3\end{aligned}$$

But, we want students to be looking for alternative approaches to solving problems. By looking for and exploring alternative solutions, students are expanding their repertoire for solving problems. This year they will encounter equations that can only be solved through alternative methods. By recognizing the equation in part (c) as equivalent to $(x - 5)^2 - 4 = 0$ (subtract four from both sides), we can use the graph to find where the parabola crosses the line $y = 0$ (the x-axis). The graph tells us the solutions are $x = 7$ and $x = 3$.

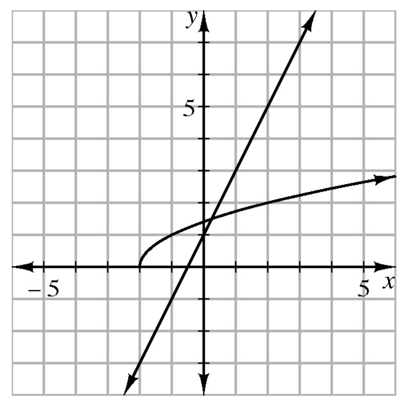
Example 2

Solve the equation $\sqrt{x+2} = 2x+1$ using at least two different approaches. Explain your methods and the implications of the solution(s).

$$\sqrt{x+2} = 2x+1$$
$$\left(\sqrt{x+2}\right)^2 = (2x+1)^2$$
$$x+2 = 4x^2+4x+1$$
$$4x^2+3x-1=0$$
$$(4x-1)(x+1)=0$$
$$x = \tfrac{1}{4},\ -1$$

Most students will begin by using algebra to solve this equation. This includes squaring both sides and solving a quadratic equation as shown at right.

A problem arises if the students do not check their work. If the students substitute each x-value back into the original equation, only one x-value checks: $x = \frac{1}{4}$. This is the only solution. We can see why the other solution does not work if we use a graph to solve the equation. The graphs of $y = \sqrt{x+2}$ and $y = 2x + 1$ are shown at right. Notice that the graphs only intersect at one point, namely $x = \frac{1}{4}$. This is the only point where they are equal. The solution to the equation is this one value; the other value is called an **extraneous** solution.



Remember that a solution is a value that makes the equation true. In our original equation, this would mean that both sides of the equation would be equal for certain values of x. Using the graphs, the solution is the x-value that has the same y-value for both graphs, or the point(s) at which the graphs intersect.

Example 3

Solve each system of equations below without graphing. For each one, explain what the solution (or lack thereof) tells you about the graph of the system.

a. $y = -\frac{2}{5}x+3$
$y = \frac{3}{5}x-2$

b. $y = -2(x-2)^2+35$
$y = -2x+15$

c. $y = \frac{1}{6}x^2 - \frac{34}{6}$
$x^2+y^2 = 25$

The two equations in part (a) are written in "$y =$" form, which makes the Substitution method the most efficient method for solving. Since both expressions in x are equal to y, we set the expressions equal to each other, and solve for x.

$$y = -\tfrac{2}{5}x+3$$
$$y = \tfrac{3}{5}x-2$$

$$y = y$$
$$-\tfrac{2}{5}x+3 = \tfrac{3}{5}x-2$$
$$5\left(-\tfrac{2}{5}x+3\right) = 5\left(\tfrac{3}{5}x-2\right)$$
$$-2x+15 = 3x-10$$
$$-5x = -25$$
$$x = 5$$

We substitute this value for x back into either one of the original equations to determine the value of y. Finally, we must check that the solution satisfies both equations.

$y = -\frac{2}{5}x + 3,\ x = 5$

$y = -\frac{2}{5}(5) + 3$

$y = -2 + 3$

$y = 1$

Solution to check: (5, 1)

Check

$y = -\frac{2}{5}x + 3$

$1 \stackrel{?}{=} -\frac{2}{5}(5) + 3$

$1 = -2 + 3$ Check

$y = \frac{3}{5}x - 2$

$1 \stackrel{?}{=} \frac{3}{5}(5) - 2$

$1 = 3 - 2$ Check

Therefore the solution is the point (5, 1), which means that the graphs of these two lines intersect in one point, the point (5, 1).

The equations in part (b) are written in the same form so we will solve this system the same way we did in part (a).

$$\begin{aligned} y &= y \\ -2(x-2)^2 + 35 &= -2x + 15 \\ -2(x-2)^2 &= -2x - 20 \\ -2(x^2 - 4x + 4) &= -2x - 20 \\ -2x^2 + 8x - 8 &= -2x - 20 \\ -2x^2 + 10x + 12 &= 0 \\ x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \\ x &= 6,\ x = -1 \end{aligned}$$

We substitute each x-value into either equation to find the corresponding y-value. Here we will use the simpler equation.

$x = 6,\ y = -2x + 15$

$y = -2(6) + 15$

$y = -12 + 15$

$y = 3$

Solution: (6, 3)

$x = -1,\ y = -2x + 15$

$y = -2(-1) + 15$

$y = 2 + 15$

$y = 17$

Solution: (–1, 17)

Lastly, we need to check each point in both equations to make sure we do not have any extraneous solutions.

$(6, 3)$: $y = -2(x-2)^2 + 35$

$3 = -2(6-2)^2 + 35$

$3 = -2(16) + 35$ Check

$(6, 3)$: $y = -2x + 15$

$3 = -2(6) + 15$ Check

$(-1, 17)$: $y = -2(x-2)^2 + 35$

$17 = -2(-1-2)^2 + 35$

$17 = -2(9) + 35$ Check

$(-1, 17)$: $y = -2x + 15$

$17 = -2(-1) + 15$ Check

In solving these two equations with two unknowns, we found two solutions, both of which check in the original equations. This means that the graphs of the equations, a parabola and a line, intersect in exactly two distinct points.

Part (c) requires substitution to solve. We can replace y in the second equation with what the first equation tells us it equals, but that will require us to solve an equation of degree four (an exponent of 4). Instead, we will first rewrite the first equation without fractions in an effort to simplify. We do this by multiplying both sides of the equation by 6.

$$y = \tfrac{1}{6}x^2 - \tfrac{34}{6}$$
$$6y = x^2 - 34$$
$$6y + 34 = x^2$$

Now we can replace the x^2 in the second equation with $6y + 34$.

$$x^2 + y^2 = 25$$
$$(6y + 34) + y^2 = 25$$
$$y^2 + 6y + 34 = 25$$
$$y^2 + 6y + 9 = 0$$
$$(y+3)(y+3) = 0$$
$$y = -3$$

Next we substitute this value back into either equation to find the corresponding x-value.

$$y = -3:\quad 6y + 34 = x^2$$
$$6(-3) + 34 = x^2$$
$$-18 + 34 = x^2$$
$$16 = x^2$$
$$x = \pm 4$$

This gives us two possible solutions: (4, –3) and (–4, –3). Be sure to check these points for extraneous solutions!

$(4,-3):\ \ y = \frac{1}{6}x^2 - \frac{34}{6},\quad -3 = \frac{1}{6}(4)^2 - \frac{34}{6} = \frac{16}{6} - \frac{34}{6} = \frac{-18}{6}$, check.

$(4,-3):\ \ x^2 + y^2 = 25,\ \ (4)^2 + (-3)^2 = 16 + 9 = 25,$ check.

$(-4,-3):\ \ y = \frac{1}{6}x^2 - \frac{34}{6},\quad -3 = \frac{1}{6}(-4)^2 - \frac{34}{6} = \frac{16}{6} - \frac{34}{6} = \frac{-18}{6}$, check.

$(-4,-3):\ \ x^2 + y^2 = 25,\ \ (-4)^2 + (-3)^2 = 16 + 9 = 25,$ check.

Since there are two points that make this system true, the graphs of this parabola and this circle intersect in only two points, (4, –3) and (–4, –3).

Example 4

Jo has small containers of lemonade and lime soda. She once mixed one lemonade container with three containers of lime soda to make 17 ounces of a tasty drink. Another time, she combined five containers of lemonade with six containers of lime soda to produce 58 ounces of another splendid beverage. Given this information, how many ounces are in each small container of lemonade and lime soda?

We can solve this problem by using a system of equations. To start, we let x equal the number of ounces of lemonade in each small container, and y equal the number of ounces of lime soda in each of its small containers. We can write an equation that describes each mixture Jo created. The first mixture used one container of x ounces of lemonade and three containers of y ounces of lime soda to produce 17 ounces. This can be represented as $1x + 3y = 17$. The second mixture used five containers of x ounces of lemonade and six containers of y ounces of lime soda to produce 58 ounces. This can be represented by the equation $5x + 6y = 58$. We can solve this system to find the values for x and y.

$$\begin{array}{lcr} x+3y=17 & \xrightarrow{\times 5} & 5x+15y=85 \\ 5x+6y=58 & \rightarrow & \underline{5x+6y=58} \\ & & 9y=27 \\ & & y=3 \end{array}$$

$$\begin{aligned} x+3(3)&=17 \\ x+9&=17 \\ x&=8 \end{aligned}$$

(Note: you should check these values!) Therefore each container of Jo's lemonade has 8 ounces, and each container of her lime soda has only 3 ounces.

Problems

Solve each of the following systems for x and y. Then explain what the answer tells you about the graphs of the equations. Be sure to check your work.

1. $x + y = 11$
   $3x - y = 5$

2. $2x - 3y = -19$
   $-5x + 2y = 20$

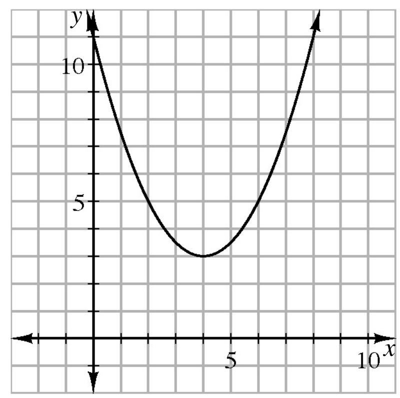
3. $15x + 10y = 21$
   $6x + 4y = 11$

4. $8x + 2y = 18$
   $-6x + y = 14$

5. $12x - 16y = 24$
   $y = \frac{3}{4}x - \frac{3}{2}$

6. $\frac{1}{2}x - 7y = -15$
   $3x - 4y = 24$

The graph of $y = \frac{1}{2}(x-4)^2 + 3$ is shown at right. Use the graph to solve each of the following equations. Explain how you get your answer.



7. $\frac{1}{2}(x-4)^2 + 3 = 3$
8. $\frac{1}{2}(x-4)^2 + 3 = 5$
9. $\frac{1}{2}(x-4)^2 + 3 = 1$
10. $\frac{1}{2}(x-4)^2 = 8$

Solve each equation below. Think about rewriting, looking inside, or undoing to simplify the process.

11. $3(x-4)^2 + 6 = 33$

12. $\frac{x}{4} + \frac{x}{5} = \frac{9x-4}{20}$

13. $3 + \left(\frac{10-3x}{2}\right) = 5$

14. $-3\sqrt{2x-1} + 4 = 10$

Solve each of the following systems of equations algebraically. What does the solution tell you about the graph of the system?

15. $y = -\frac{2}{3}x + 7$
    $4x + 6y = 42$

16. $y = (x+1)^2 + 3$
    $y = 2x + 4$

17. $y = -3(x-4)^2 - 2$
    $y = -\frac{4}{7}x + 4$

18. $x + y = 0$
    $y = (x-4)^2 - 6$

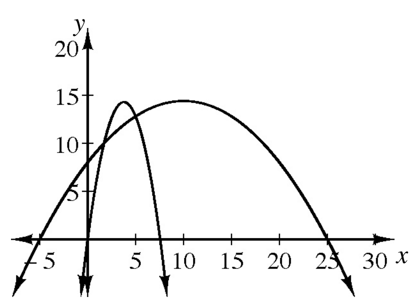
19. Adult tickets for the *Mr. Moose's Fantasy Show on Ice* are \$6.50 while a child's ticket is only \$2.50. At Tuesday night's performance, 435 people were in attendance. The box office brought in \$1667.50 for that evening. How many of each type of ticket were sold?

20. The next math test will contain 50 questions. Some will be worth three points while the rest will be worth six points. If the test is worth 195 points, how many three-point questions are there, and how many six-point questions are there?

21. Reread Example 3 from Chapter 4 about Dudley's water balloon fight. If you did this problem, you found that Dudley's water balloons followed the path described by the equation $y=-\frac{8}{125}(x-10)^2+\frac{72}{5}$. Suppose Dudley's nemesis, in a mad dash to save his base from total water balloon bombardment, ran to the wall and set up his launcher at its base. Dudley's nemesis launches his balloons to follow the path $y=-x\left(x-\frac{189}{25}\right)$ in an effort to knock Dudley's water bombs out of the air. Is Dudley's nemesis successful? Explain.

Answers

1. $(4, 7)$
2. $(-2, 5)$
3. no solution
4. $\left(-\frac{1}{2}, 11\right)$
5. All real numbers
6. $(12, 3)$
7. $x = 4$. The horizontal line $y = 3$ crosses the parabola in only one point, at the vertex.
8. $x = 2, x = 6$
9. No solution. The horizontal line $y = 1$ does not cross the parabola.
10. $x = 0, x = 8$. Add three to both sides to rewrite the equation as $\frac{1}{2}(x-4)^2+3=11$. The horizontal line $y = 11$ crosses at these two points.
11. $x = 7, x = 1$
12. no solution
13. $x = 2$
14. no solution (A square root must have a positive result.)
15. All real numbers. When graphed, these equations give the same line.
16. $(0, 4)$. The parabola and the line intersect only once.
17. No solution. This parabola and this line do not intersect.
18. $(2, -2)$ and $(5, -5)$. The line and the parabola intersect twice.
19. 145 adult tickets were sold, while 290 child tickets were sold.
20. There are 35 three-point questions and 15 six-point questions on the test.
21. By graphing we see that the nemesis' balloon when launched at the base of the wall (the y-axis), hits the path of the Dudley's water balloon. Therefore, if timed correctly, the nemesis is successful.



INEQUALITIES 4.2.1 – 4.2.4

Once the students understand the notion of a solution, they can extend their understanding to inequalities and systems of inequalities. Inequalities typically have infinitely many solutions, and students learn ways to represent such solutions. For further information see the Math Notes boxes in Lessons 4.2.2, 4.2.3, and 4.2.4.

Example 1

Solve each equation or inequality below. Explain what the solution for each one represents. Then explain how the equation and inequalities are related to each other.

$x^2 - 4x - 5 = 0$ $\qquad$ $x^2 - 4x - 5 < 0$ $\qquad$ $y \geq x^2 - 4x - 5$

Students have many ways to solve the equation, including graphing, factoring, or using the Quadratic Formula. Most students will factor and use the zero-product property to solve as shown below.

$$\begin{aligned} x^2 - 4x - 5 &= 0 \\ (x-5)(x+1) &= 0 \\ x = 5, \quad x &= -1 \end{aligned}$$

Check:

$x = 5: \quad (5)^2 - 4(5) - 5 = 25 - 20 - 5 = 0$ ✓

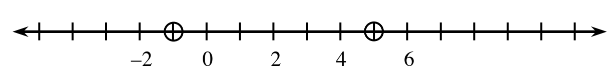
$x = -1: \quad (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0$ ✓

These are the only two values for x that make this equation true, $x = 5$ and $x = -1$.

The second quadratic is an inequality. To solve this we will utilize a number line to emphasize what the solution represents. When solving the equation, we found that the quadratic expression can be factored.

$$x^2 - 4x - 5 = (x - 5)(x + 1)$$

Using the factored form, we want to find all x-values so that $(x - 5)(x + 1) < 0$, or rephrasing it, where the product is negative. We begin by recording on a number line the places where the product equals zero. We found those two points in the previous part: $x = 5$ and $x = -1$. By placing these two points on the number line, they act as boundary points, dividing the number line into three sections. We choose any number in each of the sections to see if the number will make the inequality true or false. Solutions will make the inequality true. Note: We only need to check one point from each section. As one point goes, so goes the section! To start, choose the point $x = -2$. Substituting this into the inequality gives:



$$(-2)^2 - 4(-2) - 5 \overset{?}{<} 0$$

$$4 + 8 - 5 \overset{?}{<} 0$$

$$7 \overset{?}{<} 0$$

False! Seven is NOT less than zero, so this section is NOT part of the solution.

Next we choose a point in the middle section. An easy value to try is $x = 0$.

True! This section is part of the solution.

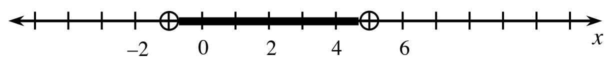
$$(0)^2 - 4(0) - 5 \overset{?}{<} 0$$
$$0 - 0 - 5 \overset{?}{<} 0$$
$$-5 \overset{?}{<} 0$$

Finally, we check to see if any points in the last section make the inequality true. Try $x = 7$.

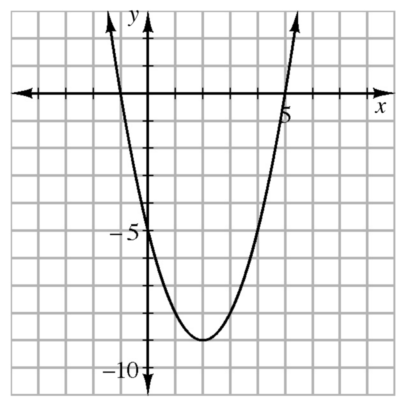
False! Therefore the solution is only the middle section, the numbers that lie between –1 and 5.

$$(7)^2 - 4(7) - 5 \overset{?}{<} 0$$
$$49 - 28 - 5 \overset{?}{<} 0$$
$$16 \overset{?}{<} 0$$

We can represent this in a couple of ways. We can use symbols to write $-1 < x < 5$. We can also represent the solution on the number line by shading the section of the number line that represents the solution of the inequality. Any point in the shaded section of the number line will make the inequality true.

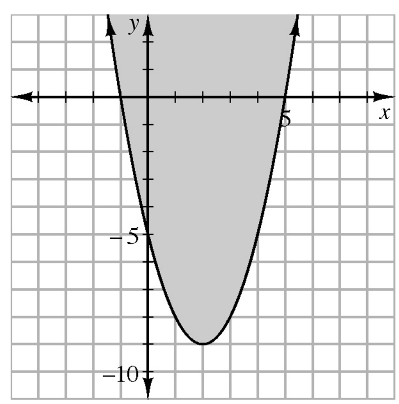


The last inequality of the example has added a y. We want to find all y-values greater than or equal to the quadratic expression. Having both x and y means we need to use an xy-coordinate graph. The graph of the parabola at right divides the plane into two regions: the part within the "bowl" of the parabola—the interior—and the region outside the parabola. The points *on* the parabola represent where $y = x^2 - 4x - 5$. We use a test point from one of the regions to check whether it will make the inequality true or false. As before, we are looking for the "true" region. The point (0, 0) is an easy point to use.



$$0 \overset{?}{\geq} (0)^2 - 4(0) - 5$$
$$0 \overset{?}{\geq} 0 - 0 - 5$$
$$0 \overset{?}{\geq} -5$$

True! Therefore the region containing the point (0, 0) is the solution. This means any point chosen in this region, the "bowl" of the parabola, will make the inequality true.



To illustrate that this region is the solution, we shade this region of the graph. Note: Since the inequality was "greater than **or equal to**," the parabola itself is included in the solution. If the inequality had been strictly "greater than," we would have made the curve dashed to illustrate that the parabola itself is not part of the solution.

To see how these equations and inequalities are related, examine the graph of the parabola. Where are the y-values of the parabola negative? Where are they equal to zero? The graph is negative when it dips below the x -axis, and this happens when x is between –1 and 5. Solving the first inequality told you that as well. It equals zero at the points $x = -1$ and $x = 5$, which you found by solving the equation. Therefore, if we had the graph initially, we could have answered the first two parts quickly by looking at the graph.

Example 2

Han and Lea have been building jet roamers and pod racers. Each jet roamer requires one jet pack and three crystallic fuel tanks, while each pod racer requires two jet packs and four crystallic fuel tanks. Han and Lea's suppliers can only produce 100 jet packs, and 270 fuel tanks each week, and due to manufacturing conditions, Han and Lea can build no more than 30 pod racers each week. Each jet roamer makes a profit of 1 tig (their form of currency) while each pod racer makes a profit of four tigs.

a. If Han and Lea receive an order for twelve jet roamers and twenty-two pod racers, how many of each part will they need to fill this order? If they can fill this order, how many tigs will they make?

b. Write a list of constraints, an inequality for each constraint, and sketch a graph showing all inequalities with the points of intersection labeled. How many jet roamers and pod racers should Han and Lea build to maximize their profits?

This problem is an example of a **linear programming** problem, and although the name might conjure up images of computer programming, these problems are not done on a computer. We solve this problem by creating a system of inequalities that, when graphed, creates a **feasibility region**. This region contains the solution for the number of jet roamers and pod racers Han and Lea should make to maximize their profit.

We begin by defining the variables. Let x represent the number of jet roamers Han and Lea will make, while y represents the number of pod racers. We know that $x \geq 0$, and $y \geq 0$. A jet roamer requires one jet pack while a pod racer requires two. There are only 100 jet packs available each week, so we can write $x + 2y \leq 100$ as one of our inequalities. Each jet roamer requires three crystallic fuel tanks and each pod racer requires four. This translates into the inequality $3x + 4y \leq 270$ since they have only 270 fuel tanks available each week. Lastly, since Han and Lea cannot make more than 30 pod racers, we can write $y \leq 30$.

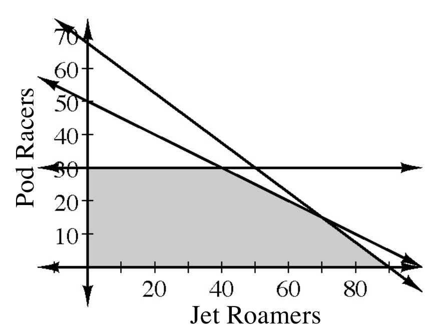
To find the number of parts needed to fill the order in part (a), we can use these inequalities with $x = 12$ and $y = 22$.

Jet packs:	Crystallic fuel tank:
$12 + 2(22) = 12 + 44 = 56$	$3(12) + 4(22) = 36 + 88 = 124$

Each result is within the constraints, so it is possible for Han and Lea to fill this order. If they do, they will make $1(12) + 4(22) = 12 + 88 = 100$ tigs.

Part (b) has us generalize this information to determine how Han and Lea can maximize their profits. Given their limited supply of parts, should they use all of them to make jet roamers?

They bring in more money per vehicle. Or, should they make some combination of the two vehicle types to ensure they use their parts and still bring in as much money as possible? We include all these inequalities on the graph of this system at right. The region common to all constraints is shaded. This is the **feasibility region** because choosing a point in this shaded area gives you a combination of jet roamers and pod racers that Han and Lea can produce under the given restraints.



To maximize profits, we will test all the vertices of this region in our profit equation, profit = $x + 4y$, to find the greatest profit. These points are: (0, 0), (0, 30), (40, 30), (70, 15), and (90, 0).

$$\text{profit: } x+4y$$

$$(0,0):\ 0+4(0)=0 \qquad (40,30):\ 40+4(30)=160$$

$$(0,30):\ 0+4(30)=120 \qquad (70,15):\ 70+4(15)=130$$

$$(90,0):\ 90+4(0)=90$$

The greatest profit is 160 tigs when Han and Lea build 40 jet roamers and 30 pod racers.

Problems

Graph the following system of inequalities. Shade the solution region (the region containing the points that satisfy all of the inequalities).

1. $y < \frac{1}{2}x + 6$
   $y > -\frac{1}{2}x + 6$
   $x < 12$

2. $x + y < 10$
   $x + y > 4$
   $y < 2x$
   $y > 0$

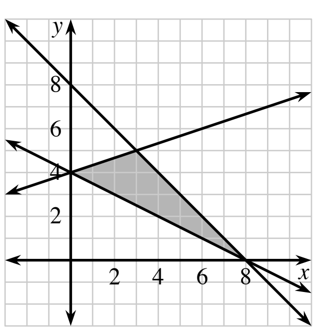
3. $y \le 3x + 4$
   $y \ge -\frac{1}{4}x + 8$
   $y \ge -\frac{1}{3}x + 4$
   $y \ge 5x - 6$

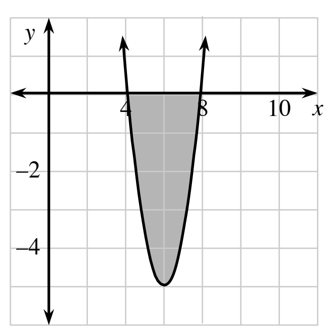
4. $3x + 4y < 12$
   $y > (x + 1)^2 - 4$

5. $y < -\frac{3}{4}(x-1)^2 + 6$
   $y > x - 7$

6. $y < (x + 2)^3$
   $y > x^2 + 3x$

Write a system of inequalities that when graphed will produce these regions.

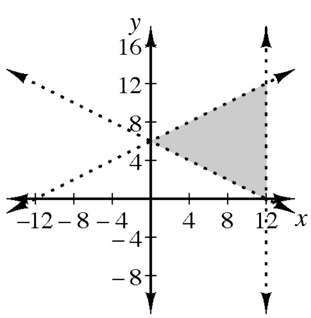
7. 

8. 

9. Ramon and Thea are considering opening their own business. They wish to make and sell alien dolls they call *Hauteans* and *Zotions*. Each *Hautean* sells for $1.00 while each *Zotion* sells for $1.50. They can make up to 20 *Hauteans* and 40 *Zotions*, but no more than 50 dolls total. When Ramon and Thea go to city hall to get a business license, they find there are a few more restrictions on their production. The number of *Zotions* (the more expensive item) can be no more than three times the number of *Hauteans* (the cheaper item). How many of each doll should Ramon and Thea make to maximize their profit? What will the profit be?

10. Sam and Emma are plant managers for the *Sticky Chewy Candy Company* that specializes in delectable gourmet candies. Their two most popular candies are *Chocolate Chews* and *Peanut and Jelly Jimmies*. Each batch of *Chocolate Chews* takes 1 teaspoon of vanilla while each batch of the *Peanut and Jelly Jimmies* uses two teaspoons of vanilla. They have at most 20 teaspoons of vanilla on hand as they use only the freshest of ingredients. The *Chocolate Chews* use two teaspoons of baking soda while the *Peanut and Jelly Jimmies* use three teaspoons of baking soda. They only have 36 teaspoons of baking soda on hand. Because of production restrictions, they can make no more than 15 batches of *Chocolate Chews* and no more than 7 batches of *Peanut and Jelly Jimmies*. Sam and Emma have been given the task of determining how many batches of each candy they should produce if they make $3.00 profit for each batch of *Chocolate Chews* and $2.00 for each batch of *Peanut and Jelly Jimmies*. Help them out by writing the inequalities described here, graphing the feasibility region, and determining their maximum profit.

Answers

1.


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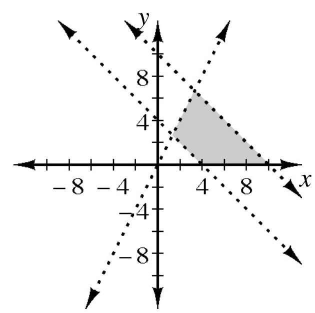

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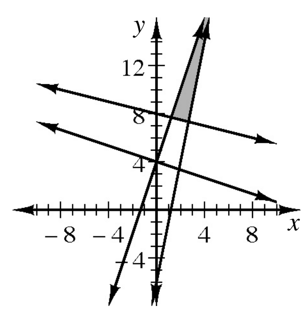

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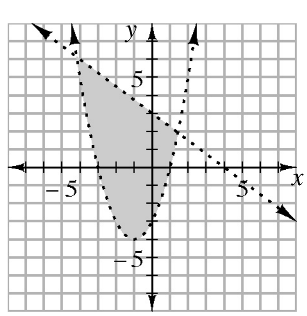

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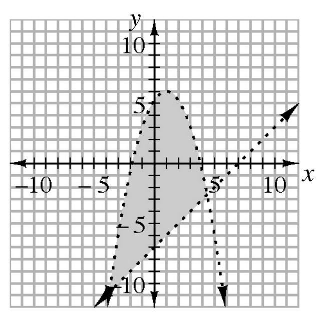

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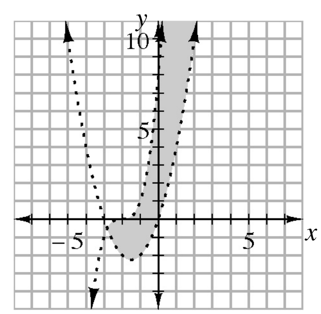


7. $y \le \frac{1}{3}x + 4$

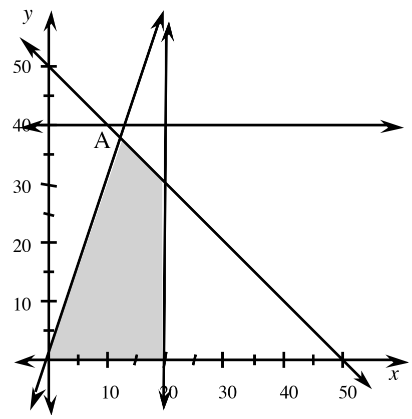
$y \le -x + 8$

$y \ge -\frac{1}{2}x + 4$

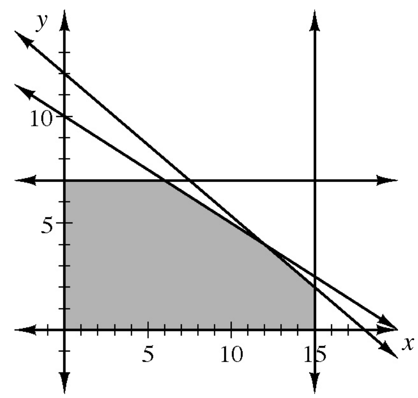
8. $y \ge (x-6)^2 - 5$

$y \le 0$

9. The graph of the feasibility region is shown at right. The inequalities are $x \ge 0$, $y \ge 0$, $x + y \le 50$, $x \le 20$, $y \le 40$, and $y \le 3x$, where x is the number of *Hauteans* and y is the number of *Zotions*. The profit is given by $P = x + 1.5y$. Maximum profit seems to occur at point A (12.5, 37.5), but there is a problem with this point. Ramon and Thea cannot make a half of a doll (or at least that does not seem possible). Try these nearby points: (12, 37), (12, 38), (13, 37), and (13, 38). The point that gives maximum profit *and* is still in the feasibility region is (13, 37). They should make 13 *Hautean* and 37 *Zotion* dolls for a profit of \$68.50.



10. The graph of the feasibility region is shown at right. The inequalities are $x \ge 0$, $y \ge 0$, $y \le 7$, $x \le 15$, $x + 2y \le 20$, and $2x + 3y \le 36$, where x = number of *Chocolate Chews* and y = number of *Peanut and Jelly Jimmies*. The profit is given by $P = 3x + 2y$. The point that seems to give the maximum profit is (15, 2.5) but this only works if half batches can be made. Instead, choose the point (15, 2) which means Sam and Emma should make 15 batches of *Chocolate Chews* and 2 batches of *Peanut and Jelly Jimmies*. Their profit will be \$49.00.



SAT PREP

1. If $\frac{x+4}{12} = \frac{4}{3}$, then x equals:

   a. 3 b. 6 c. 8 d. 10 e. 12

2. What is the least of three consecutive integers whose sum is 21?

   a. 5 b. 6 c. 7 d. 8 e. 9

3. Juanita has stocks, bonds, and t-bills for investments. The number of t-bills she has is one more than the number of stocks, and the number of bonds is three times the number of t-bills. Which of the following could be the total number of investments?

   a. 16 b. 17 c. 18 d. 19 e. 20

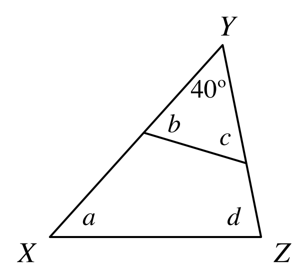
4. Through how many degrees would the minute hand of a clock turn from 5:20 p.m. to 5:35 p.m. the same day?

   a. 15° b. 30° c. 45° d. 60° e. 90°

5. The length of a rectangle is six times its width. If the perimeter of the rectangle is 56, what is the width of the rectangle?

   a. 4 b. 7 c. 8.5 d. 18 e. 24

6. If $m > 1$ and $m^n m^5 = m^{15}$, then what does n equal?



7. In the triangle at right, what is the value of $a + b + c + d$?

8. If x and y are positive integers, $x + y < 12$, and $x > 4$, what is the greatest possible value for $x - y$?

9. If $(2x^2 + 5x + 3)(2x + 4) = ax^3 + bx^2 + cx + d$ for all values of x, what does c equal?

10. Four lines intersect in one point creating eight congruent adjacent angles. What is the measure of one of these angles?

Answers

1. E
2. B
3. D
4. E
5. A
6. 10
7. 280°
8. 9
9. 26
10. 45°

SAT PREP

1. If $\frac{x+4}{12} = \frac{4}{3}$, then x equals:

   a. 3 b. 6 c. 8 d. 10 e. 12

2. What is the least of three consecutive integers whose sum is 21?

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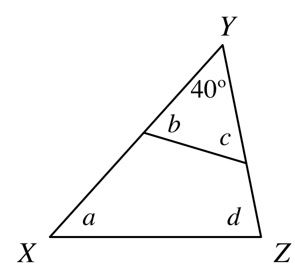

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INVERSES 5.1.1 – 5.1.3

Students explore inverses, that is, equations that "undo" the actions of functions. Although they may not be aware of it, students have already seen inverses without calling them that. When solving equations, students reverse operations to undo the equation leaving just x, and that is what inverses do. Students further explore composition of functions by considering what happens when inverses are combined. For further information see the Math Notes boxes in Lessons 5.1.2 and 5.1.3.

Example 1

Find the inverse (undo) rule for the functions below. Use function notation and give the inverse rule a name different from the original function.

a. $f(x) = \frac{x-6}{3}$

b. $g(x) = (x + 4)^2 + 1$

The function in part (a) subtracts 6 from the input then divides by 3. The undo rule, or inverse, reverses this process. Therefore, the inverse first multiplies by 3 then adds 6. If we call this inverse $h(x)$, we can write $h(x) = 3x + 6$.

The function $g(x)$ adds 4 to the input, squares that value, then adds 1. The inverse will first subtract 1, take the square root then subtract 4. Calling this rule $j(x)$ we can write $j(x) = \pm\sqrt{x-1} - 4$.

Rather than give the inverse a new name, we can use the notation for inverses. The inverse of $f(x)$ is written as $f^{-1}(x)$.

Note: The inverses $h(x)$ and $j(x)$, are fundamentally different. $h(x) = 3x + 6$ is the equation of a non-vertical line, therefore $h(x)$ is a function. $j(x) = \pm\sqrt{x-1} - 4$, however, is not a function. By taking the square root, we created a positive value and a negative value. This gives two outputs for each input, and so by definition it is *not* a function.

Although we computed the inverses through a verbal description of what each function does, the students learn an algorithm for finding an inverse. They switch the x and y, and then solve for y. Using this algorithm on the equations above:

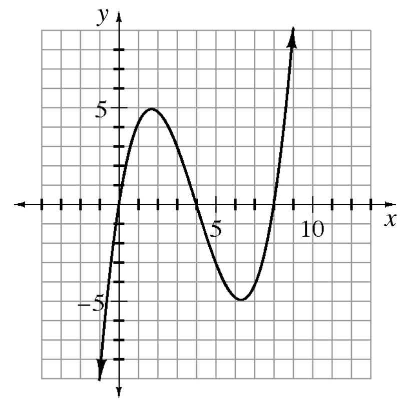
$$\begin{aligned} f(x) &= \tfrac{x-6}{3} \\ y &= \tfrac{x-6}{3} \\ x &= \tfrac{y-6}{3} \\ 3x &= y - 6 \\ 3x + 6 &= y \end{aligned}$$

$$\begin{aligned} g(x) &= (x+4)^2 + 1 \\ y &= (x+4)^2 + 1 \\ x &= (y+4)^2 + 1 \\ x - 1 &= (y+4)^2 \\ \pm\sqrt{x-1} &= y + 4 \\ -4 \pm \sqrt{x-1} &= y \end{aligned}$$

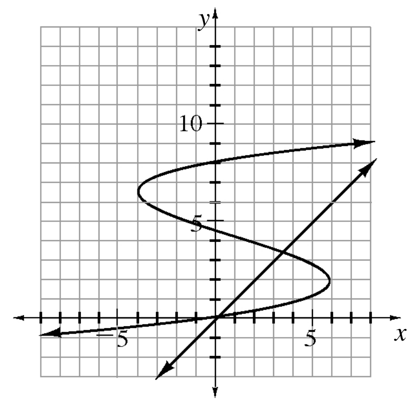
Example 2

The graph of $f(x) = 0.2x^3 - 2.4x^2 + 6.4x$ is shown at right. Graph the inverse of this function.

Following the algorithm for determining the equation for an inverse, as we did above, would be difficult here. The students do not have a method for solving cubic equations. Nevertheless, students can graph the inverse because they know a special property about the graphs of functions and their inverses: they are symmetrical about the line $y = x$.



If we add the line $y = x$ to the graph, the inverse is the reflection across this line. Here we can fold the paper along the line $y = x$, and trace the result to create the reflection.



Example 3

Consider the function $f(x) = \frac{2x-1}{7}$. Determine the inverse of $f(x)$ and label it $g(x)$. Verify that these two functions are inverses by calculating $f\left(g(x)\right)$ and $g\left(f(x)\right)$.

Using the algorithm, we can determine the inverse.

$$y = \frac{2x-1}{7}$$
$$x = \frac{2y-1}{7}$$
$$7x = 2y - 1$$
$$7x + 1 = 2y$$
$$y = \frac{7x+1}{2}$$
$$g(x) = \frac{7x+1}{2}$$

Composing the two functions $f(x)$ and $g(x)$ gives a method for checking whether or not functions are inverses of each other. Since one function "undoes" the other, when the functions are composed, the output should be x.

$$f(x) = \frac{2x-1}{7} \qquad g(x) = \frac{7x+1}{2}$$

$$f\left(g(x)\right) = f\left(\frac{7x+1}{2}\right) = \frac{2\left(\frac{7x+1}{2}\right)-1}{7} = \frac{7x+1-1}{7} = \frac{7x}{7} = x$$

$$g\left(f(x)\right) = g\left(\frac{2x-1}{7}\right) = \frac{7\left(\frac{2x-1}{7}\right)+1}{2} = \frac{2x-1+1}{2} = \frac{2x}{2} = x$$

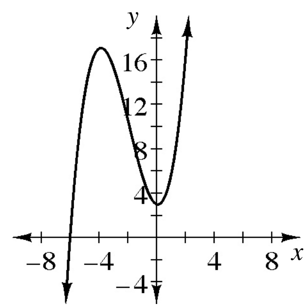
Since $f\left(g(x)\right) = g\left(f(x)\right) = x$, the functions are inverses.

Problems

Find the inverse of each of the following functions.

1. $f(x) = 8(x - 13)$
2. $y = -\frac{3}{4}x + 6$
3. $y = \frac{5(x+2)}{3}$
4. $f(x) = x^2 + 6$
5. $f(x) = \frac{3}{x} + 6$
6. $g(x) = \frac{5}{x}$
7. $g(x) = (x + 1)^2 - 3$
8. $y = (x + 2)^3$
9. $y = 3 + \sqrt{x - 4}$
10. $g(x) = 6x + 2$

Sketch the graph of the inverse of each of the following functions.

11. $y = \frac{x}{6} + 2$
12. $f(x) = 2x^2 - 1$
13. $g(x) = x$
14. $y = \frac{1}{5x}$
15. 

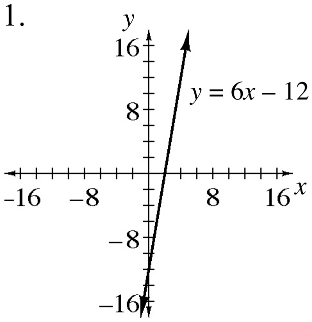
For each of the following pairs of functions, determine $f(g(x))$ and $g(f(x))$, then use the result to decide whether or not $f(x)$ and $g(x)$ are inverses of each other.

16. $f(x) = 5x + 7$
    $g(x) = \frac{x-7}{5}$
17. $f(x) = 8x$
    $g(x) = \frac{1}{8}x$
18. $f(x) = x + 5$
    $g(x) = \frac{1}{x+5}$
19. $f(x) = \frac{2}{3x}$
    $g(x) = \frac{3x}{2}$
20. $f(x) = \frac{2}{3}x + 6$
    $g(x) = \frac{3(x-6)}{2}$
21. $f(x) = x\sqrt{3} + 9$
    $g(x) = \left(\frac{x-9}{\sqrt{3}}\right)^2$

Answers

1. $y = \frac{x}{8} + 13$
2. $y = -\frac{4}{3}x + 8$
3. $y = \frac{3}{5}x - 2$
4. $y = \pm\sqrt{x-6}$
5. $y = \frac{3}{x-6}$
6. $y = \frac{5}{x}$
7. $y = -1 \pm \sqrt{x+3}$
8. $y = -2 + \sqrt[3]{x}$
9. $y = (x-3)^2 + 4$, for $x \geq 3$
10. $y = \frac{x-2}{6}$

11.



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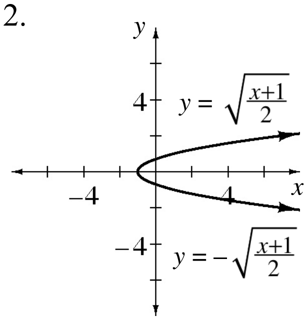

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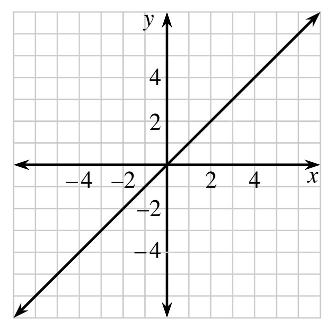

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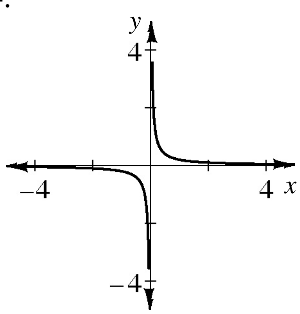

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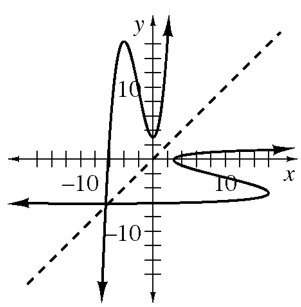


16. $f(g(x)) = g(f(x)) = x$. They are inverses.
17. $f(g(x)) = g(f(x)) = x$. They are inverses.
18. $f(g(x)) = \frac{1}{x+5} + 5$, $g(f(x)) = \frac{1}{x+10}$. No, they are not inverses.
19. $f(g(x)) = \frac{4}{9x}$, $g(f(x)) = \frac{1}{x}$. No, they are not inverses.
20. $f(g(x)) = g(f(x)) = x$. They are inverses.
21. $f(g(x)) = \frac{\sqrt{3}(x-9)^2}{3} + 9$, $g(f(x)) = x^2$. No, they are not inverses.

LOGARITHMS 5.2.1 – 5.2.4

The earlier sections of this chapter gave students many opportunities to find the inverses of various functions. Here, students explore the inverse of an exponential function. Although they can graph the inverse by reflecting the graph of an exponential function across the line $y = x$, they cannot write the equation of this new function. Writing the equation requires the introduction of a new function, the logarithm. Students explore the properties and graphs of logarithms, and in a later chapter use them to solve equations of this type. For further information see the Math Notes box in Lesson 5.2.2.

Example 1

Find each of the values below and then justify your answer by writing the equivalent exponential form.

a. $\log_5(25) = ?$ b. $\log_7(?) = 3$ c. $\log_2\left(\frac{1}{8}\right) = ?$

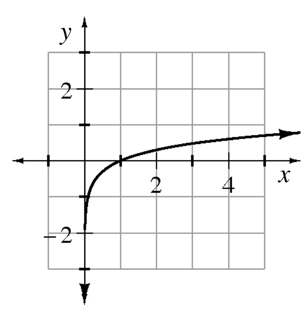
A logarithm is really just an exponent, so an expression like the one in part (a), $\log_5(25)$, is asking "What exponent can I raise the base 5 to, to get 25?" We can translate this question into an equation: $5^? = 25$. By phrasing it this way, the answer is more apparent: 2. This is true because $5^2 = 25$.

Part (b) can be rephrased as $7^3 = ?$. The answer is 343.

Part (c) asks "2 to what exponent gives $\frac{1}{8}$?" or $2^? = \frac{1}{8}$. The answer is –3 because $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

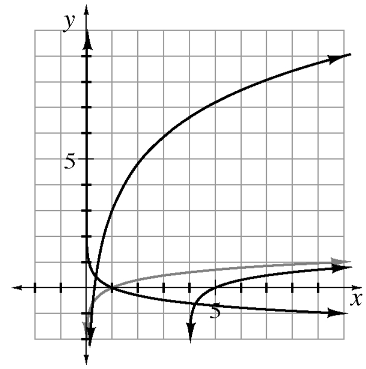
Example 2

The graph of $y = \log(x)$ is shown at right. Use this "parent graph" to graph each of the following equations. Explain how you get your new graphs.



$y = \log(x - 4)$ $y = 6\log(x) + 3$ $y = -\log(x)$

The logarithm function follows the same rules for transforming its graphs as other functions we have used. The first equation shifts the original graph to the right four units. The graph of the second equation is shifted up three units (because of the "+ 3") but is also stretched because it is multiplied by six. The third function is flipped across the x-axis. All three of these graphs are shown at right. The original function $y = \log(x)$ is also there, in light gray. Note: When a logarithm is written without a base, as in $y = \log(x)$ and the log key used on a calculator, the base is 10.



Problems

Rewrite each logarithmic equation as an exponential equation and vice versa.

1. $y = \log_4(x)$
2. $3 = \log_2(x)$
3. $x = \log_5(30)$
4. $4^x = 80$
5. $\left(\frac{1}{2}\right)^x = 64$
6. $x^3 = 343$
7. $5^x = \frac{1}{125}$
8. $\log_x(32) = y$
9. $11^3 = x$
10. $-4 = \log_x\left(\frac{1}{16}\right)$

What is the value of x in each equation below? If necessary, rewrite the expression in the equivalent exponential equation to verify your answer.

11. $4 = \log_5(x)$
12. $2 = \log_9(x)$
13. $9 = \log(x)$
14. $81 = 9^x$
15. $\left(\frac{1}{3}\right)^x = 243$
16. $6^x = 7776$
17. $7^x = \frac{1}{49}$
18. $\log_2(32) = x$
19. $\log_{11}(x) = 3$
20. $\log_5\left(\frac{1}{125}\right) = x$

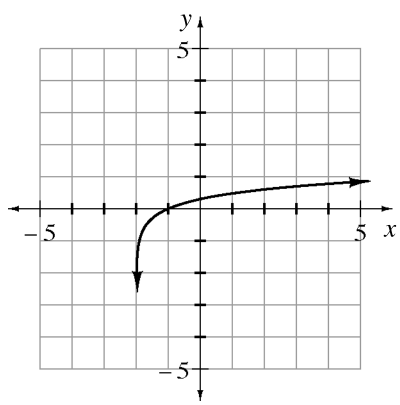
Graph each of the following equations.

21. $y = \log(x + 2)$
22. $y = -5 + \log(x)$
23. $y = -\log(x - 4)$
24. $y = 5 + 3\log(x - 7$

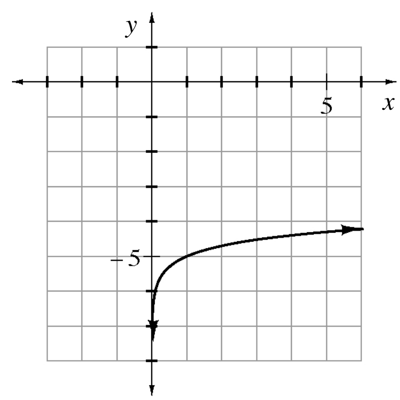
Answers

1. $4^y = x$
2. $2^3 = x$
3. $5^x = 30$
4. $\log_4(80) = x$
5. $\log_{1/2}(64) = x$
6. $\log_x(343) = 3$
7. $\log_5\left(\frac{1}{125}\right) = x$
8. $x^y = 32$
9. $\log_{11}(x) = 3$
10. $x^{-4} = \frac{1}{16}$
11. $x = 625$
12. $x = 81$
13. $x = 1{,}000{,}000{,}000$
14. $x = 2$
15. $x = -5$
16. $x = 5$
17. $x = -2$
18. $x = 5$
19. $x = 1331$
20. $x = -3$

21.



22.



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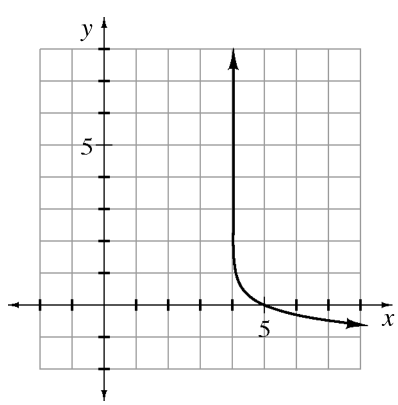

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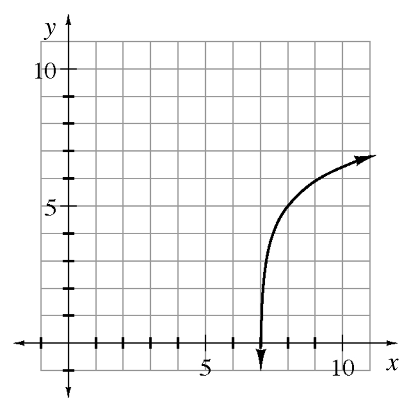
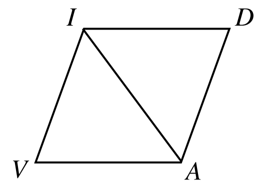


SAT PREP

1. In the figure at right, ΔDAI is isosceles with $DI = 13$ and base 24. If $\Delta VAI \cong \Delta DAI$ what is the area of quadrilateral $DAVI$?

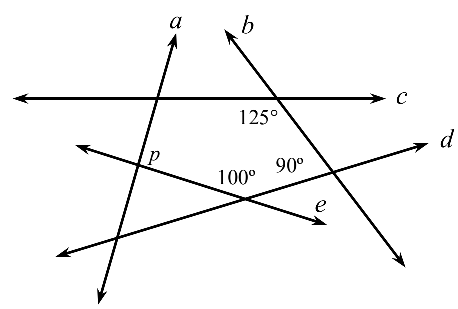


a. 60 b. 75 c. 120 d. 156 e. 240

2. An experimental jet flies at a speed of 5280 miles per hour. How many miles can this jet cover in 10 seconds?

a. 1.467 b. 8.802 c. 11.237 d. 14.667 e. 88.022

3. If the angle (not shown) where a and b intersect is three times as large as the angle (not shown) where e and b intersect, what is the value of p?



a. 70° b. 85° c. 140°

d. 160° e. Cannot determine

4. Let $\zeta x \zeta$ be defined for all positive integer values of x as the product of all even factors of $4x$. For example, $\zeta 3 \zeta = 12 \times 6 \times 4 \times 2 = 576$. What is the value of $\zeta 5 \zeta$?

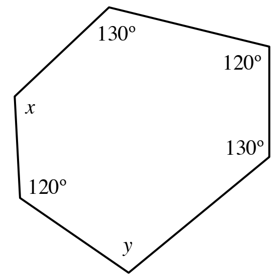
a. 1600 b. 6400 c. 7200 d. 8000 e. 9600

5. The chart at right shows the distribution of topics covered in a particular business text, in chapters and pages per chapter. According to the chart, how many total pages are in this text?

Topic	No. of Chapters	No. of pages
Development	3	12
Marketing	4	8
Public Relations	1	11

a. 31 b. 39 c. 48 d. 65 e. 79

6. In the figure at right, what is the sum of x and y?
Note: The figure is not drawn to scale.



7. If $2^q = 8^{q-1}$, then q = ?

8. If a is 40 percent of 300, b is 40 percent of a, and c is 25 percent of b, what is $a + b + c$?

9. If $\frac{x}{4} = \frac{11}{20}$, what is the value of x?

10. If $\frac{3}{5}$ of $\frac{1}{3}$ is added to 5, what is the answer?

Answers

1. C
2. D
3. C
4. A
5. E
6. 220°
7. $q = \frac{3}{2}$
8. 180
9. $x = 2.2$
10. $5\frac{1}{5}$

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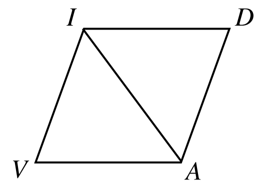

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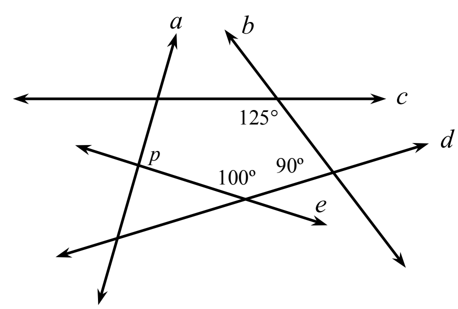

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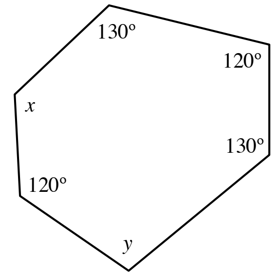

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Answers

1. C
2. D
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6. 220°
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8. 180
9. $x = 2.2$
10. $5\frac{1}{5}$

SOLVING WITH LOGARITHMS 6.2.1 and 6.2.4

Students turn their attention back to logarithms. Using Guess and Check, Pattern Recognition, and other problem solving strategies, students develop several properties of logs that enable them to solve equations that have been, until now, very cumbersome to solve. These properties are listed in the Math Notes box in Lesson 6.2.2.

Example 1

Solve each of the following equations for x.

a. $5^x = 67$

b. $3(7^x) + 4 = 124$

Each of these problems has the variable as the exponent, which makes them different from others that students have been solving. So far, students have been solving problems similar to these by Guess and Check. This approach has been time consuming and difficult to find an accurate answer.

Now students can use the log property, $\log(b^x) = x\log(b)$, to solve these equations for x. As with other equations, however, students must isolate the variable on one side of the equation. Note: The decimal answer is an approximation. The exact answer is the fraction $\frac{\log(67)}{\log(5)}$.

$$\begin{aligned} 5^x &= 67 \\ \log(5^x) &= \log(67) \\ x\log(5) &= \log(67) \\ x &= \frac{\log(67)}{\log(5)} \\ x &\approx 2.61252 \end{aligned}$$

Some work must be done to the second equation before we can incorporate logs. We will move everything we can to one side of the equation so that the variable is as isolated as possible (steps 1 through 3).

$$\begin{aligned} 3(7^x)+4 &= 124 \\ 3(7^x) &= 120 \\ 7^x &= 40 \\ \log(7^x) &= \log(40) \\ x\log(7) &= \log(40) \\ x &= \frac{\log(40)}{\log(7)} \\ x &\approx 1.89571 \end{aligned}$$

Example 2

Using the properties of logs of products and quotients, rewrite each product as a sum, each quotient as a difference, and vice versa.

a. $\log_3(16x) =$ 		b. $\log_6(32) + \log_6(243) =$

c. $\log_8\left(\frac{3x}{7}\right) =$ 		d. $\log_{12}(276) - \log_{12}(23) =$

The two properties we will use are $\log(ab) = \log(a) + \log(b)$ and $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$. These properties are true for any base, so we can use the first one to rewrite part (a) as $\log_3(16x) = \log_3(x)$. This new form is not necessarily better or simpler, it is just another way to represent the expression. In part (b), we can use the first property to write $\log_6(32) + \log_6(243) = \log_6(32 \cdot 243) = \log_6(7776)$. Although it is not necessary, this can be simplified further. Since $6^5 = 7776$, $\log_6(7776) = 5$.

We will rewrite parts (c) and (d) using the second property listed above. Therefore, $\log_8\left(\frac{3x}{7}\right) = \log_8(3x) - \log_8(7)$. Note: We could use the first property to expand this further by writing $\log_8(3x)$ as $\log_8(3) + \log_8(x)$. Working in the opposite direction on part (d), we write $\log_{12}(276) - \log_{12}(23) = \log_{12}\left(\frac{276}{23}\right)$. Simplifying further, $\log_{12}\left(\frac{276}{23}\right) = \log_{12}(12) = 1$.

Example 3

Fall came early in Piney Orchard, and the community swimming pool was still full when the first frost froze the leaves. The outside temperature hovered at 30º. Maintenance quickly turned off the heat so that energy would not be wasted heating a pool that nobody would be swimming in for at least six months. As Tess walked by the pool each day on her way to school, she would peer through the fence at the slowly cooling pool. She could just make out the thermometer across the deck that displayed the water's temperature. On the first day, she noted that the water temperature was 68º. Four days later, the temperature reading was 58º. Write an equation that models this data. If the outside temperature remains at 30º, and the pool is allowed to cool, how long before it freezes?

Heating and cooling problems are typical application problems that use exponential equations. In class, students solved such a problem, *The Case of the Cooling Corpse*. The equation that will model this problem is an exponential equation of the form $y = km^x + b$. In the problem description, we are given two data points: (0, 68º) and (4, 58º). We also have another piece of important information. The outside temperature is hovering at 30°. This is the temperature the water will approach, that is, $y = 30$ is the horizontal asymptote for this equation. Knowing this fact allows us to write the equation as $y = km^x + b$. To determine k and m, we will substitute our values into the equation and solve for k and m.

$$(0,68) \Rightarrow y = km^x + 30 \Rightarrow 68 = km^0 + 30$$
$$(4,58) \Rightarrow y = km^x + 30 \Rightarrow 58 = km^4 + 30$$

This gives us two equations with two unknowns that we can solve. Simplifying first makes our work a lot easier. The first equation simplifies to $38 = k$ since $m^0 = 1$. Since $k = 38$ we can substitute this value into the second equation to determine m.

$$\begin{aligned}58 &= km^4 + 30\\ 58 &= 38m^4 + 30\\ 28 &= 38m^4\\ m^4 &= \tfrac{28}{38} \approx 0.7368\\ m &\approx 0.9265\end{aligned}$$

Therefore the equation is $y = 38(0.9265)^x + 30$. To determine when the pool will freeze, we want to find when the water's temperature reaches 32º.

$$\begin{aligned}32 &= 38(0.9265)^x + 30\\ 2 &= 38(0.9265)^x\\ \tfrac{2}{38} &= 0.9265^x\\ \log\left(\tfrac{2}{38}\right) &= \log(0.9265^x)\\ \log\left(\tfrac{2}{38}\right) &= x\log(0.9265)\\ x &= \frac{\log\left(\frac{2}{38}\right)}{\log(0.9265)} \approx 38.57\end{aligned}$$

In approximately 38.5 days, the water in the pool will freeze if the outside temperature remains at 30º for those days. In reality, the pool would be drained to prevent damage from freezing.

Problems

Solve each of the following equations for x.

1. $(2.3)^x = 7$
2. $12^x = 6$
3. $\log_7 49 = x$
4. $\log_3 x = 4$
5. $5(3.14)^x = 18$
6. $7x^8 = 294$
7. $\log_x 100 = 4$
8. $\log_5 45 = x$
9. $2(6.5)^x + 7 = 21$
10. $-\frac{1}{2}(14)^x + 6 = -9.1$

Rewrite each log of a product as a sum of logs, each difference of logs as a log of a quotient, and vice versa.

11. $\log(23 \cdot 3)$
12. $\log\left(\frac{3x}{8}\right)$
13. $\log_2\left(\frac{60}{7}\right)$
14. $\log_8(12) - \log_8(2)$
15. $\log_5(25) + \log_5(25)$
16. $\log(10 \cdot 10)$
17. $\log_{13}(15x^2)$
18. $\log(123) + \log(456)$
19. $\log(10^8) - \log(10^3)$
20. $\log(5x - 4)$

Simplify.

21. $\log_2(64)$
22. $\log_{17}(17^{1/8})$
23. $8^{\log_8(1.3)}$
24. $2.3^{5\log_{2.3}(1)}$

25. Climbing Mt. Everest is not an easy task! Not only is it a difficult hike, but the Earth's atmosphere decreases exponentially as you climb above the Earth's surface, and this makes it harder to breathe. The air pressure at the Earth's surface (sea level) is approximately 14.7 pounds per square inch (or 14.7 psi). In Denver, Colorado, elevation 5280 feet, the air pressure is approximately 12.15 psi. Write the particular equation representing this data expressing air pressure as a function of altitude. What is the air pressure in Mexico City, elevation 7300 feet? At the top of Mt. Everest, elevation 29,000 feet? (Note: You will need to carry out the decimal values several places to get an accurate equation and air pressures.)

Answers

1. $x = \frac{\log(7)}{\log(2.3)} \approx 2.336$
2. $x = \frac{\log(6)}{\log(12)} \approx 0.721$
3. $x = 2$
4. $x = 81$
5. $x = \frac{\log(3.6)}{\log(3.14)} \approx 1.119$
6. $x \approx 1.596$
7. $x \approx 3.162$
8. $x = \frac{\log(45)}{\log(5)} \approx 2.365$
9. $x = \frac{\log(7)}{\log(6.5)} \approx 1.040$
10. $x = \frac{\log(30.2)}{\log(14)} \approx 1.291$
11. $\log(23) + \log(3)$
12. $\log(3x) - \log(8)$
13. $\log_2(60) - \log_2(7)$
14. $\log_8\left(\frac{12}{2}\right) = \log_8(6)$
15. $\log_5(625)$
16. $\log(10) + \log(10)$
17. $\log_{13}(15) + \log_{13}(x^2)$
18. $\log(56{,}088)$
19. $\log\left(\frac{10^8}{10^3}\right) = \log 10^5$
20. Already simplified.
21. 6
22. $\frac{1}{8}$
23. 1.3
24. 1
25. The particular equation is $y = 1.47(0.999964)^x$ where x is the elevation, and y is the number of pounds per square inch (psi). The air pressure in Mexico City is approximately 11.3 psi, and at the top of Mt. Everest, the air pressure is approximately 5.175 psi.

SAT PREP

1. If $b + 4 = 11$, then $(b - 2)^2 =$

a. 16 b. 25 c. 36 d. 49 e. 64

2. Let P and Q represent digits in the addition problem shown at right. What must the digit Q be?

$$\begin{array}{r} 25P \\ +P4 \\ \hline 32Q \end{array}$$

a. 0 b. 1 c. 2 d. 3 e. 4

3. If $3^4 = 9^x$, then $x =$

a. 2 b. 3 c. 5 d. 8 e. 10

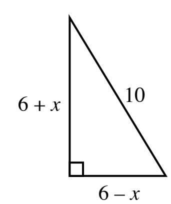
4. When a positive number n is divided by 7 the remainder is 6. Which of the following expressions will yield a remainder of 1 when it is divided by 7?

a. $n + 1$ b. $n + 2$ c. $n + 3$ d. $n + 4$ e. $n + 5$

5. How many 4-digit numbers have the thousands digit equal to 2 and the units digit equal to 7?

a. 100 b. 199 c. 200 d. 500 e. 10005

6. In the figure at right, where $x < 6$, what is the value of $x^2 + 36$?



a. 10 b. 50 c. 100

d. 600 e. 1296

7. The measures of the angles of a triangle in degrees can be expressed by the ratio 5:6:7. What is the sum of the measures of the two larger angles?

a. 110 b. 120 c. 130 d. 160 e. 180

8. If $\frac{r}{3} = \frac{7}{10}$, what is the value of r ?

9. If p and q are two different prime numbers greater than 2, and $n = pq$, how many positive factors, including 1 and n, does n have?

10. If $\frac{1}{2}\left(30x^2 + 20x^2 + 10x + 1\right) = ax^3 + bx^2 + cx + d$, for all values of x where a, b, c, and d are all constants, what is the value of $a + b + c + d$?

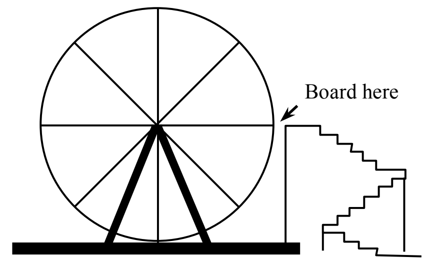
Answers

1. B
2. A
3. A
4. B
5. A
6. B
7. C
8. $r = 2.1$
9. 4
10. 30.5

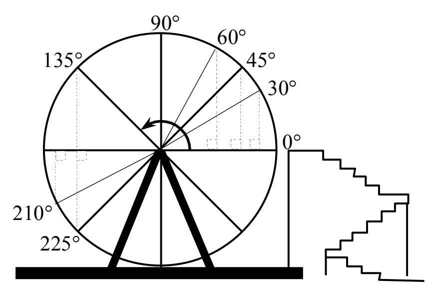
TRIGONOMETRIC FUNCTIONS 7.1.1 – 7.1.7

This chapter extends the students' knowledge of trigonometry. Students have already studied right triangle trigonometry, using sine, cosine and tangent with their calculators to find the lengths of unknown sides of triangles. Now students explore these same three trigonometry terms as functions. They are introduced to the unit circle, and they explore how the trigonometric functions are found within the unit circle. In addition, they learn a new way to measure angles using radian measure. For further information see the Math Notes boxes in Lessons 7.1.2, 7.1.5, 7.1.6, and 7.1.7.

Example 1



As Daring Davis stands in line waiting to ride the huge Ferris wheel, he notices that this Ferris wheel is not like any of the others he has ridden. First, this Ferris wheel does not board the passengers at the lowest point of the ride; rather, they board after climbing several flights of stairs, at the level of that wheel's horizontal axis. Also, if Davis thinks of the boarding point as a height of zero above that axis, then the maximum height above the boarding point that a person rides is 25 feet, and the minimum height below the boarding point is –25 feet. Use this information to create a graph that shows how a passenger's height on the Ferris wheel depends on the number of degrees of rotation from the boarding point of the Ferris wheel.



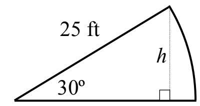
As the Ferris wheel rotates counterclockwise, a passenger's height above the horizontal axis increases, and reaches its maximum of 25 feet above the axis after 90° of rotation. Then the passenger's height decreases as measured from the horizontal axis, reaching zero feet after 180° of rotation, and continues to decrease as measured from the horizontal axis. The minimum height, –25 feet, occurs when the passenger has rotated 270°. After rotating 360°, the passenger is back where he started, and the ride continues.

To create this graph, we calculate the height of the passenger at various points along the rotation. These heights are shown using the grey line segments drawn from the passenger's location on the wheel perpendicular to the horizontal axis of the Ferris wheel. Note: Some of these values are easily filled in. At 0°, the height above the axis is zero feet. At 90°, the height is 25 feet.

Rotation, Degrees	0°	30°	45°	60°	90°	135°	180°	210°	225°	270°	315°	360°
Height, Feet	0				25		0			–25		0

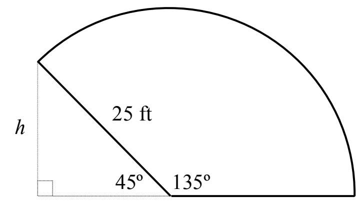
To complete the rest of the table we calculate the heights using right triangle trigonometry. We will demonstrate three of these values, 30°, 135°, and 225°, and allow you to verify the rest.

Each of these calculations involves focusing on the portion of the picture that makes a right triangle. For the 30º point, we look at the right triangle with a hypotenuse of 25 feet. (The radius of the circle is 25 feet because it is the maximum and minimum height the passenger reaches.) In this right triangle, we can use the sine function:



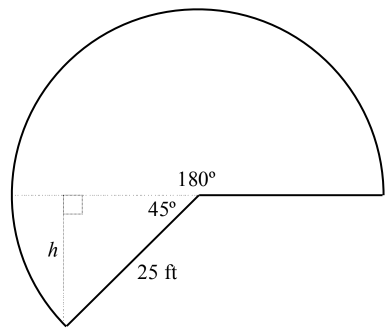
$$\sin(30^\circ) = \frac{h}{25}$$
$$25\sin(30^\circ) = h$$
$$h = 12.5 \text{ feet}$$

At the 135º mark, we use the right triangle on the "outside" of the curve. Since the angles are supplementary, the angle we use measures 45º.



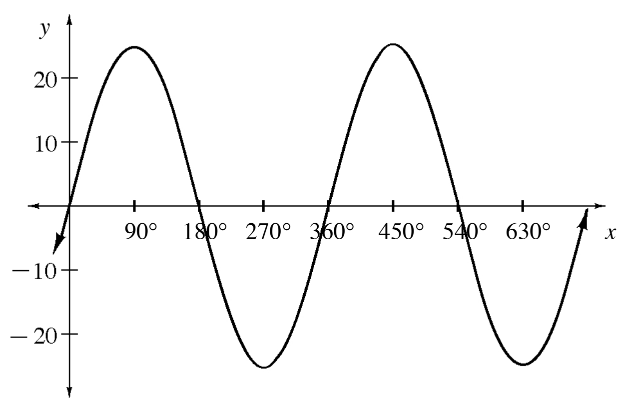
$$\sin(45^\circ) = \frac{h}{25}$$
$$25\sin(45^\circ) = h$$
$$h \approx 17.68 \text{ feet}$$

At 225º (225 = 180 + 45), the triangle we use drops below the horizontal axis. We will use the 45º angle that is within the right triangle, so $h \approx -17.68$, using the previous calculation and changing the sign to represent that the rider is below the starting point. Now we can fill in all the values of the table.



Rotation, Degrees	0º	30º	45º	60º	90º	135º	180º	210º	225º	270º	315º	360º
Height, Feet	0	12.5	17.68	21.65	25	17.68	0	–12.5	–17.68	–25	–17.68	0

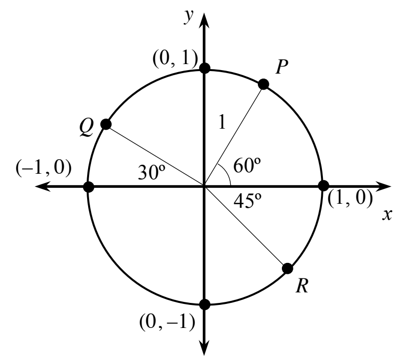
Plot these points and connect them with a smooth curve; your graph should look like the one at right. Note: This curve shows two revolutions of the Ferris wheel. This curve continues, repeating the cycle for each revolution of the Ferris wheel. It also represents a particular sine function: $y = 25\sin(x)$.



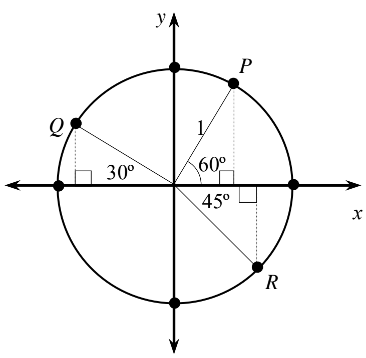
Example 2

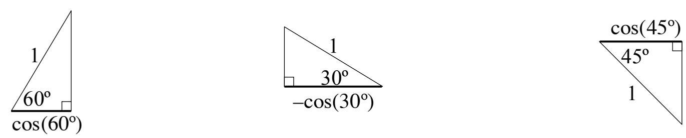
On a unit circle, represent and then calculate cos(60º), cos(150º), and cos(315º). Then graph $y = \cos(x)$.

The trigonometric functions ("trig" functions) arise naturally in circles as we saw with the first example. The simplest circle is a unit circle, that is, a circle of radius 1 unit, and it is this circle we often use with the trig functions.

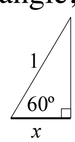


On the unit circle at right, several points are labeled. Point P corresponds to a 60º rotation, point Q corresponds to 150º, and R corresponds to 315º. We measure rotations from the point (1, 0) counter-clockwise to determine the angle. If we create right triangles at each of these points, we can use the right triangle trig we learned in geometry to determine the lengths of the legs of the triangle. In the previous example, the height of the triangle was found using the sine. Here, the cosine will give us the length of the other leg of the triangle.





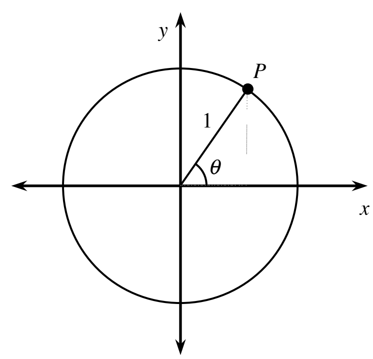
To fully understand why the length of the horizontal leg is labeled with "cosine," consider the triangle below. In the first triangle, if we labeled the short leg x, we would write:



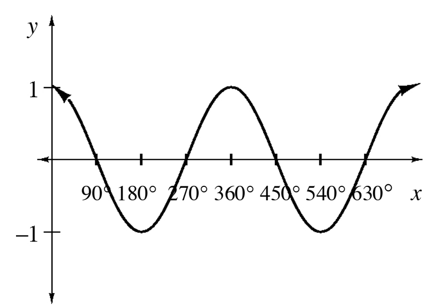
$$\cos(60^\circ) = \frac{x}{1}$$

$$x = \cos(60^\circ)$$

Therefore the length of the horizontal leg of the first triangle is cos(60º). Note: The second triangle representing 150º, lies in the second quadrant where the x-values are negative. Therefore cos(150º) = –cos(30º). Check this on your calculator.



It is important to note what this means. On a unit circle, we can find a point P by rotating θ degrees. If we create a right triangle by dropping a height from point P to the x-axis, the length of this height is always $\sin(\theta)$. The length of the horizontal leg is always $\cos(\theta)$. Additionally, this means that the coordinates of point P are $(\cos(\theta), \sin(\theta))$. This is the power of using a unit circle: the coordinates of any point on the circle are found by taking the sine and cosine of the angle. The graph at right shows the cosine curve for two rotations around the unit circle.

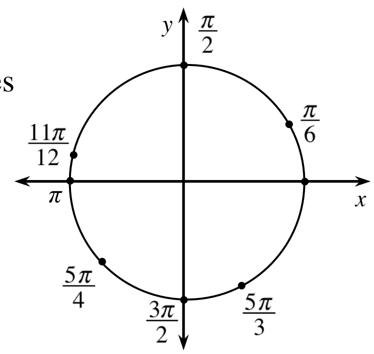


Example 3

On a unit circle, find the points corresponding to the following radians. Then convert each angle given in radians to degrees.

a. $\frac{\pi}{6}$ b. $\frac{11\pi}{12}$ c. $\frac{5\pi}{4}$ d. $\frac{5\pi}{3}$

One radian is about 57°, but that is not the way to remember how to convert from degrees to radians. Instead, think of the unit circle, and remember that one rotation would be the same as traveling around the unit circle one circumference. The circumference of the unit circle is $C = 2\pi r = 2\pi(1) = 2\pi$. Therefore, one rotation around the circle, 360°, is the same as traveling 2π radians around the circle. Radians do not just apply to unit circles. A circle with any size radius still has 2π radians in a 360° rotation.



We can place these points around the unit circle in appropriate places without converting them. First, remember that 2π radians is the same point as a 360° rotation. That makes half of that, 180°, corresponds to π radians. Half of that, 90°, is $\frac{\pi}{2}$ radians. With similar reasoning, 270° corresponds to $\frac{3\pi}{2}$ radians. Using what we know about fractions allows us to place the other radian measures around the circle. For example, $\frac{\pi}{6}$ is one-sixth the distance to π.

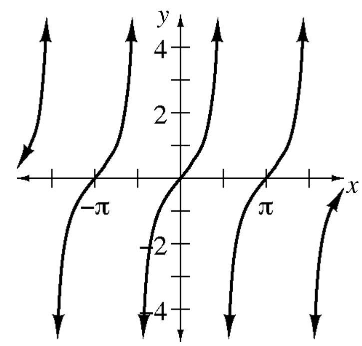
Sometimes we want to be able to convert from radians to degrees and back. To do so, we can use a ratio of $\frac{\text{radians}}{\text{degrees}}$. To convert $\frac{\pi}{6}$ radians to degrees we create a ratio, and solve for x. We will use $\frac{\pi}{180}$ as a simpler form of $\frac{2\pi}{360}$. Therefore $\frac{\pi}{6}$ radians is equivalent to 30°. Similarly, we can convert the other angles above to degrees:

$$\frac{\pi}{180°} = \frac{\pi/6}{x}$$
$$x\pi = 180\left(\frac{\pi}{6}\right)$$
$$x\pi = 30\pi$$
$$x = 30°$$

$$\frac{\pi}{180°} = \frac{11\pi/12}{x} \qquad \frac{\pi}{180°} = \frac{5\pi/4}{x} \qquad \frac{\pi}{180°} = \frac{5\pi/3}{x}$$
$$x\pi = 180°\left(\frac{11\pi}{12}\right) \qquad x\pi = 180°\left(\frac{5\pi}{4}\right) \qquad x\pi = 180°\left(\frac{5\pi}{3}\right)$$
$$x\pi = 165°\,\pi \qquad x\pi = 225°\,\pi \qquad x\pi = 300°\,\pi$$
$$x = 165° \qquad x = 225° \qquad x = 300°$$

Example 4

Graph $T(\theta) = \tan(\theta)$. Explain what happens at the points $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$. Why does this happen?



As with the graphs of $S(\theta) = \sin(\theta)$ and $C(\theta) = \cos(\theta)$, $T(\theta) = \tan(\theta)$ repeats, that is, it is cyclic. The graph does not, however, have the familiar hills and valleys the other two trig functions display. This graph, shown at right, resembles in part the graph of a cubic such as $f(x) = x^3$. However, it is *not* a cubic, which is clear from the fact that it has asymptotes and repeats. At $\theta = \frac{\pi}{2}$, the graph approaches a vertical asymptote. This also occurs at $\theta = -\frac{\pi}{2}$, and because the graph is cyclic, it happens repeatedly at $\theta = \frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, … . In fact, it happens at all values of θ of the form $\frac{(2k-1)\pi}{2}$ for all integer values of k (odd values).

The real question is, *why* does this asymptote occur at these points? Recall that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. Every point where $\cos(\theta) = 0$, this function is undefined (we cannot have zero in the denominator). So at each point where $\cos(\theta) = 0$, the function $T(\theta) = \tan(\theta)$ is also undefined. Examining the graph of $C(\theta) = \cos(\theta)$, we can see that this graph is zero (crosses the x-axis) at the same type of points as above: $\frac{(2k-1)\pi}{2}$ for all integer values of k.

Problems

Graph each of the following trig equations.

1. $y = \sin(x)$
2. $y = \cos(x)$
3. $y = \tan(x)$

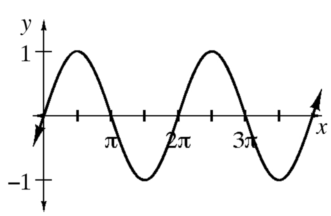
Find each of the following values without using a calculator, but by using what you know about right triangle trigonometry, the unit circle, and special right triangles.

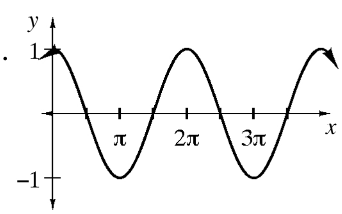
4. $\cos(180°)$
5. $\sin(360°)$
6. $\tan(45°)$
7. $\cos(-90°)$
8. $\sin(150°)$
9. $\tan(240°)$

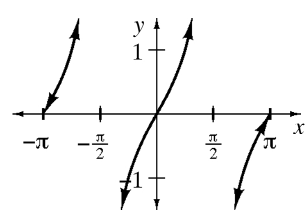
Convert each of the angle measures.

10. $60°$ to radians
11. $170°$ to radians
12. $315°$ to radians
13. $\frac{\pi}{15}$ radians to degrees
14. $\frac{13\pi}{8}$ radians to degrees
15. $-\frac{3\pi}{4}$ radians to degrees

Answers

1. 

2. 

3. 

4. -1
5. 0
6. 1
7. 0
8. $\frac{1}{2}$
9. $\sqrt{3}$
10. $\frac{\pi}{3}$ radians
11. $\frac{17\pi}{18}$ radians
12. $\frac{7\pi}{4}$ radians
13. $12°$
14. $292.5°$
15. $-135°$

TRANSFORMING TRIG FUNCTIONS 7.2.1 – 7.2.4

Students apply their knowledge of transforming parent graphs to the trigonometric functions. They will generate general equations for the family of sine, cosine and tangent functions, and learn about a new property specific to cyclic functions called the period. The Math Notes box in Lesson 7.2.4 illustrates the different transformations of these functions.

Example 1

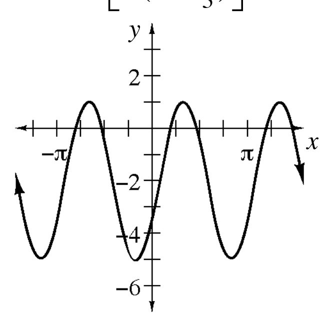
For each of the following equations, state the amplitude, number of cycles in 2π, horizontal shift, and vertical shift of the graph. Then graph each equation on a separate set axes.

$$y = 3\cos\left[2(x - \tfrac{\pi}{3})\right] - 2 \qquad\qquad y = -\sin\left[\tfrac{1}{4}(x + \pi)\right] + 1$$

The general form of a sine function is $y = a\sin[b(x - h)] + k$. Some of the transformations of trig functions are standard ones that students learned in Chapter 2. The a will determine the orientation, in this case, whether it is in the standard form, or if it has been reflected across the x-axis. With trigonometric functions, a also represents the amplitude of the function: half of the distance the function stretches from the maximum and minimum points vertically. As before, h is the horizontal shift, and k is the vertical shift. This leaves just b, which tells us about the period of the function. The graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ each have a period of 2π, which means that one cycle (before it repeats) has a length of 2π. However, b affects this length since b tells us the number of cycles that occur in the length 2π.

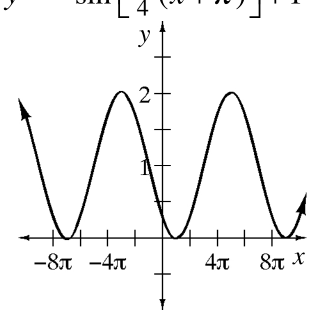
The first function, then, has an amplitude of 3, and since this is positive, it is not reflected across the x-axis. The graph is shifted horizontally to the right $\frac{\pi}{3}$ units, and shifted down (vertically) 2 units. The 2 before the parentheses tells us it does two cycles in 2π units. If the graph does two cycles in 2π units, then the length of the period is π units. The graph of this function is shown at right.

$y = 3\cos\left[2(x - \tfrac{\pi}{3})\right] - 2$



The second function has an amplitude of 1, but it is reflected across the x-axis. It is shifted to the left π units, and shifted up 1 unit. Here we see that within a 2π span, only one fourth of a cycle appears. This means the period is four times as long as normal, or is 8π. The graph is shown at right.

$y = -\sin\left[\tfrac{1}{4}(x + \pi)\right] + 1$



Example 2

For the Fourth of July parade, Vicki decorated her tricycle with streamers and balloons. She stuck one balloon on the outside rim of one of her back tires. The balloon starts at ground level. As she rides, the height of balloon rises up and down, sinusoidally (that is, a sine curve). The diameter of her tire is 10 inches.

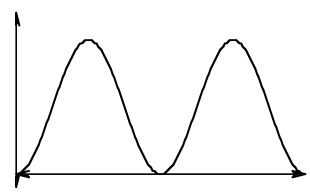
a. Sketch a graph showing the height of the balloon above the ground as Vicki rolls along.

b. What is the period of this graph?

c. Write the equation of this function.

d. Use your equation to predict the height of the balloon after Vicki has traveled 42 inches.

This problem is similar to the Ferris Wheel example at the beginning of this chapter. The balloon is rising up and down just as a sine or cosine curve rises up and down. A simple sketch is shown at right.

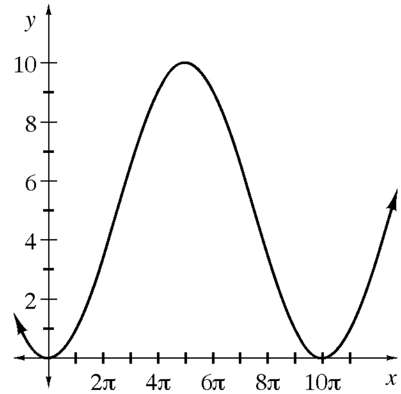


The balloon begins next to the ground and as the tricycle wheel rolls, the balloon rises to the top of the wheel, then comes back down. If we let the ground represent the x-axis, the balloon is at its highest point when it is at the top of the wheel, a distance of one wheel's height or diameter, 10 inches. So now we know that the distance from the highest point to the lowest point is 10. The amplitude is half of this distance, 5.

To determine the period, we need to think about the problem. The balloon starts at ground level, rises as the wheel rolls and comes down again to the ground. What has happened when the balloon returns to the ground? The wheel has made one complete revolution. How far has the wheel traveled then? It has traveled the distance of one circumference. The circumference of a circle with diameter 10 inches is 10π inches. Therefore the period of this graph is 10π.

To get the equation for this graph we need to make some decisions. The graphs of sine and cosine are similar. In fact, one is just the other shifted 90° (or $\frac{\pi}{2}$ radians). At this point, we need to decide if we want to use sine or cosine to model this data. Either one will work but the answers will look different. Since the graph starts at the lowest point and not in the middle, this suggests that we use cosine. (Yes, cosine starts at the highest point but we can multiply by a negative to flip the graph over and start at the lowest point.) We also know the amplitude is 5 and there is no horizontal shift. All of this information can be written in the equation as $y = -5\cos(bx) + k$. We can determine k by remembering that we set the x-axis as the ground. This implies the graph is shifted up 5 units. To determine the number of cycles in 2π (that is, b), recall that we found that the period of this graph is 10π. Therefore $\frac{2\pi}{10} = \frac{1}{5}$ of the curve appears within the 2π span. Finally, pulling everything together we can write $y = -5\cos\left[\frac{1}{5}x\right] + 5$, and is shown in the following graph.

$$y = -5\cos\left[\tfrac{1}{5}x\right] + 5$$



Note: If you decided to use the sine function for this data, you must realize that the graph is shifted to the right $\frac{10\pi}{4}$ units. One equation that gives this graph is $y = 5\sin\left[\frac{1}{5}(x - \frac{10\pi}{4})\right] + 5$. There are other equations that work, so if you do not get the same equation as shown here, graph yours and compare.

To find the height of the balloon after Vicki rides 42 inches, we substitute 42 for x in the equation.

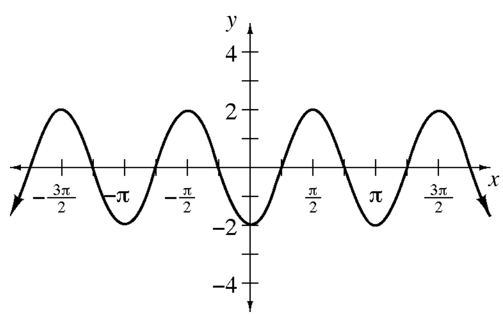
If you do not get this answer, make sure your calculator is in radians!

$$\begin{aligned} y &= -5\cos\left[\tfrac{1}{5}\cdot 42\right] + 5 \\ &\approx -5\cos(8.4) + 5 \\ &\approx 7.596 \text{ inches} \end{aligned}$$

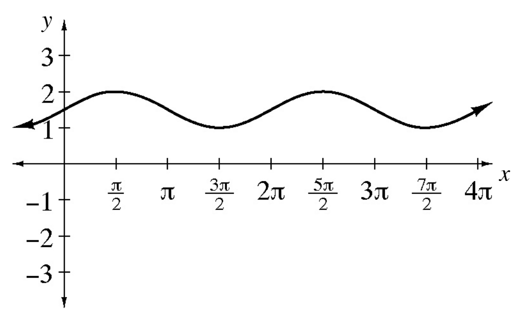
Problems

Examine each graph below. For each one, draw a sketch of one cycle, then give the amplitude and the period.

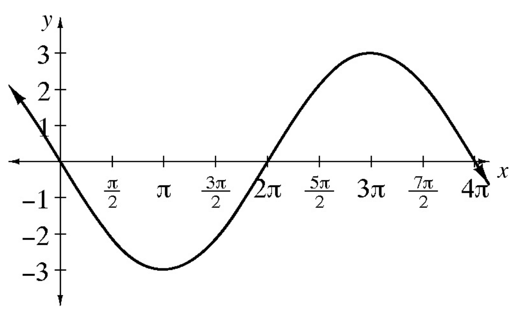
1.



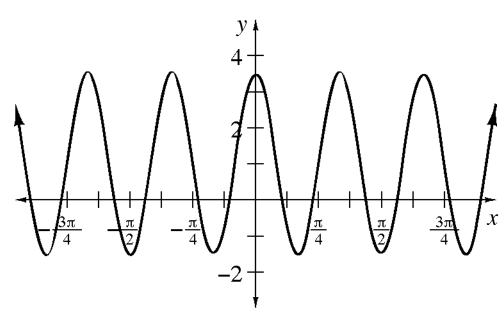
2.



3.



4.



For each equation listed below, state the amplitude and period.

5. $y = 2\cos(3x) + 7$

6. $y = \frac{1}{2}\sin(x) - 6$

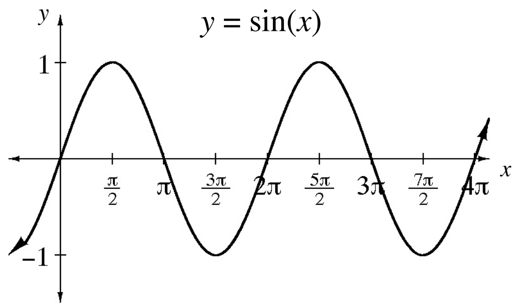
7. $f(x) = -3\sin(4x)$

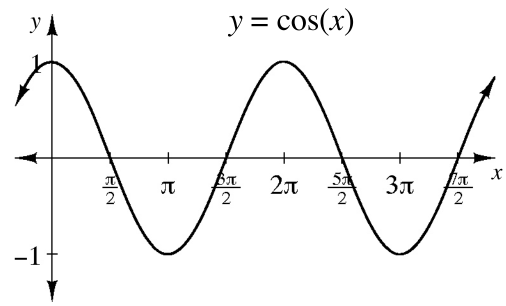
8. $y = \sin\left[\frac{1}{3}x\right] + 3.5$

9. $f(x) = -\cos(x) + 2\pi$

10. $f(x) = 5\cos(x-1) - \frac{1}{4}$

Below are the graphs of $y = \sin(x)$ and $y = \cos(x)$.





Use them to sketch the graphs of each of the following equations and functions by hand. Use your graphing calculator to check your answer.

11. $y = -2\sin(x + \pi)$

12. $f(x) = \frac{1}{2}\sin(3x)$

13. $f(x) = \cos\left(4\left(x - \frac{\pi}{4}\right)\right)$

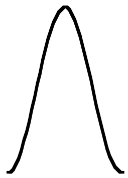
14. $y = 3\cos\left(x + \frac{\pi}{4}\right) + 3$

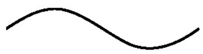
15. $f(x) = 7\sin\left(\frac{1}{4}x\right) - 3$

16. A wooden water wheel is next to an old stone mill. The water wheel makes ten revolutions every minute, dips down two feet below the surface of the water, and at its highest point is 18 feet above the water. A snail attaches to the edge of the wheel when the wheel is at its lowest point and rides the wheel as it goes round and around. As time passes, the snail rises up and down, and in fact, the height of the snail above the surface of the water varies sinusoidally with time. Use this information to write the particular equation that gives the height of the snail over time.

17. To keep baby Cristina entertained, her mother often puts her in a Johnny Jump Up. It is a seat on the end of a strong spring that attaches in a doorway. When Mom puts Cristina in, she notices that the seat drops to just 8 inches above the floor. One and a half seconds later (1.5 seconds), the seat reaches its highest point of 20 inches above the ground. The seat continues to bounce up and down as time passes. Use this information to write the particular equation that gives the height of baby Cristina's Johnny Jump Up seat over time. (Note: You can start the graph at the point where the seat is at its lowest.)

Answers

1.  Amplitude is 2, period is π.

2.  Amplitude 0.5, period 2π.

3. The graph shows one cycle already. Amplitude is 3 and period is 4π.

4.  Amplitude is 2.5, period is $\frac{\pi}{3}$.

5. Amplitude: 2, period: $\frac{2\pi}{3}$.

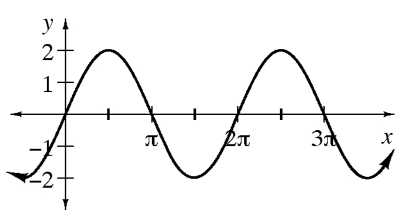
6. Amplitude: $\frac{1}{2}$, period: 2π.

7. Amplitude: 3, period: $\frac{\pi}{2}$.

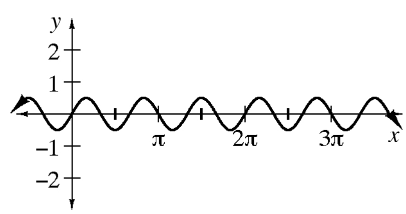
8. Amplitude: 1, period: 6π.

9. Amplitude: 1, period: 2π.

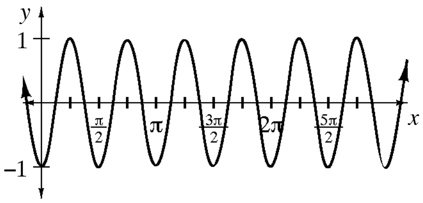
10. Amplitude: 5, period: 2π.

11. 

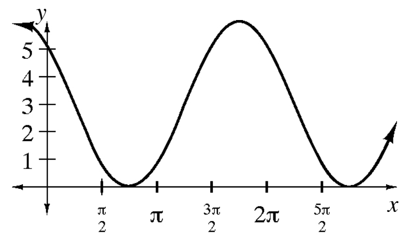
Surprised? The negative flips it over, but the "+ π" shifts it right back to how it looks originally!

12. 

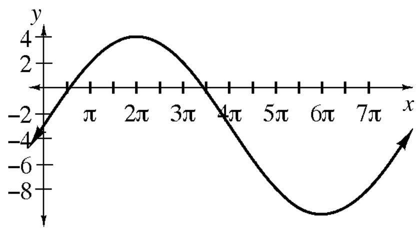
13.



14.



15.



16. $y = -10\cos\left(\frac{1}{10}x\right) + 8$, and there are other possible equations which will work.

17. $y = -6\cos\left(\frac{2\pi}{3}x\right)$ works if we let the graph be symmetric about the *x*-axis. The *x*-axis does not have to represent the ground. If you let the *x*-axis represent the ground, you equation might look like $y = -6\cos\left(\frac{2\pi}{3}x\right) + 14$.

SAT PREP

1. If one "*pentaminute*" is the same as five minutes of time, how many *pentaminutes* are equivalent to four hours of time?

   a. 1200 b. 240 c. 60 d. 48 e. 20

2. If $a = 12$ and $b = -4$, what is the value of $4a - 3b$?

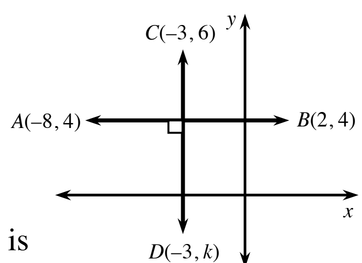
   a. 60 b. 36 c. 16 d. 9 e. –52

3. The average (arithmetic mean) of 4 and s is equal to the average of 3, 8 and s. What does s equal?

   a. 3 b. 5.5 c. 9 d. 10 e. No such s exists.

4. In the figure at right, $AB = CD$. What does k equal?

   a. –6 b. –5 c. –4
   d. –3 e. –2



5. The initial term of a sequence is 36. Each term after that is half of the term before it *if* that term is even. If the preceding term is odd, the next term is one half that term, plus one half. What is the sixth term of this sequence?

   a. 1 b. 2.25 c. 2 d. 3.5 e. 4

6. At a spa, the customer is offered a choice of five different massages and eight different pedicures. How many different combinations are there of one massage and one pedicure?

   a. 3 b. 13 c. 16 d. 28 e. 40

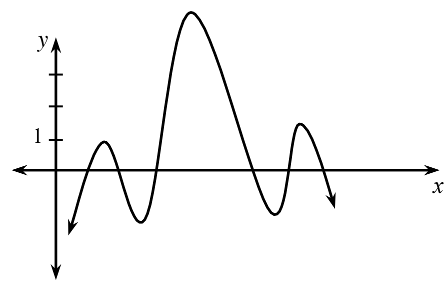
7. A rectangular box is 12 cm long, 20 cm wide, and 15 cm high. If exactly 60 smaller identical rectangular boxes can be stored perfectly in this larger box, which of the following could be the dimensions, in cm, of these smaller boxes?

   a. 5 by 6 by 12 b. 4 by 5 by 6 c. 3 by 5 by 6
   d. 3 by 4 by 6 e. 2 by 5 by 6

8. When Harry returned his book to the library, Madame Pince told him he owed a fine of \$6.45. This included \$3.00 for three weeks, plus a fine of \$0.15 per day for every day he was late in returning the book. How many overdue days did Harry have the book?

9. What is the slope of the line that passes through the points (0, 2) and (–10, –2)?

10. At right is the complete graph of the function $f(x)$. For how many positive values of x does $f(x) = 3$?



Answers

1. D
2. A
3. D
4. C
5. C
6. E
7. E
8. 23 days
9. $\frac{2}{5}$
10. 2

SAT PREP

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   a. 60 b. 36 c. 16 d. 9 e. –52

3. The average (arithmetic mean) of 4 and s is equal to the average of 3, 8 and s. What does s equal?

   a. 3 b. 5.5 c. 9 d. 10 e. No such s exists.

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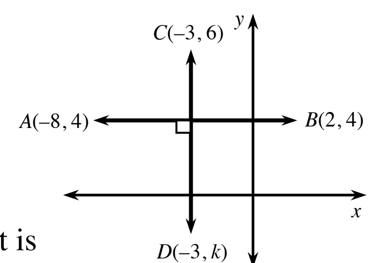
   d. –3 e. –2



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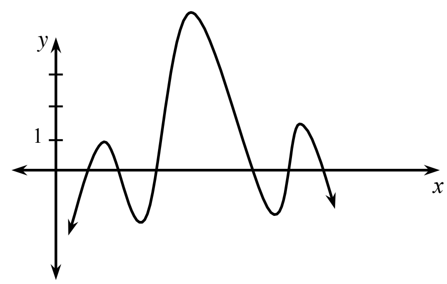
   d. 3 by 4 by 6 e. 2 by 5 by 6

8. When Harry returned his book to the library, Madame Pince told him he owed a fine of \$6.45. This included \$3.00 for three weeks, plus a fine of \$0.15 per day for every day he was late in returning the book. How many overdue days did Harry have the book?

9. What is the slope of the line that passes through the points (0, 2) and (–10, –2)?

10. At right is the complete graph of the function $f(x)$. For how many positive values of x does $f(x) = 3$?



Answers

1. D
2. A
3. D
4. C
5. C
6. E
7. E
8. 23 days
9. $\frac{2}{5}$
10. 2

POLYNOMIALS 8.1.1 – 8.1.3

The chapter explores polynomial functions in greater depth. Students will learn how to sketch polynomial functions without using their graphing tool by using the factored form of the polynomial. In addition, they learn the reverse process: finding the polynomial equation from the graph. For further information see the Math Notes boxes in Lessons 8.1.1, 8.1.2, and 8.1.3.

Example 1

State whether or not each of the following expressions is a polynomial. If it is not, explain why not. If it is a polynomial, state the degree of the polynomial.

a. $-7x^4 + \frac{2}{3}x^3 + x^2 - 4.1x - 6$

b. $8 + 3.2x^2 - \pi x^5 - 61x^{10}$

c. $9x^3 + 4x^2 - 6x^{-1} + 7^x$

d. $x(x^3 + 2)(x^4 - 4)$

A polynomial is an expression that can be written as the sum or difference of terms. The terms are in the form ax^n where a is any number called the coefficient of x, and n, the exponent, must be a whole number. The expression in part (a) is a polynomial. A coefficient that is a fraction $\left(\frac{2}{3}\right)$ is acceptable. The degree of the polynomial is the largest exponent on the variable, so in this case the degree is four. The expression in part (b) is also a polynomial, and its degree is ten. The expression in part (c) is not a polynomial for two reasons. First, the x^{-1} is not allowed because the exponents of the variable cannot be negative. The second reason is because of the 7^x. The variable cannot be a power in a polynomial. Although the expression in part (d) is not the sum or difference of terms, it can be written as the sum or difference of terms by multiplying the expression and simplifying. Doing this gives $x^8 + 2x^5 - 4x^4 - 8x$, which is a polynomial of degree 8.

Example 2

Without using your graphing tool, make a sketch of each of the following polynomials by using the orientation, roots, and degree.

a. $f(x) = (x + 1)(x - 3)(x - 4)$

b. $y = (x - 6)^2(x + 1)$

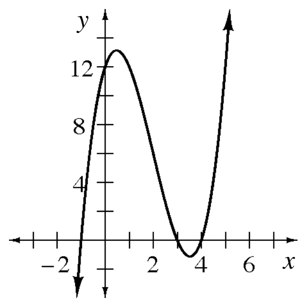
c. $p(x) = x(x + 1)^2(x - 4)^2$

d. $f(x) = -(x + 1)^3(x - 1)^2$

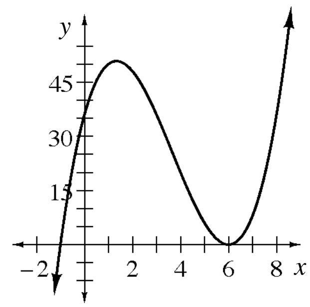
Through investigations, students learn a number of things about the graphs of polynomial functions. The roots of the polynomial are the x-intercepts, which are easily found when the polynomial is in factored form, as are all the polynomials above. Ask yourself the question: what values of x will make this expression equal to 0? The answer will give you the roots. In part (a), the roots of this third degree polynomial are $x = -1, 3$, and 4. In part (b), the roots of this third degree polynomial are 6 and -1. The degree of a polynomial tells you the maximum number of roots possible, and since this third degree polynomial has just two roots, you might ask where is the third root? $x = 6$ is called a double root, since that expression is squared and is thus equivalent to $(x-6)(x-6)$. The graph will just touch the x-axis at $x = 6$, and "bounce" off. The fifth degree polynomial in part (c) has three roots, 0, -1, and 4 with both -1 and 4 being double roots. The fifth degree polynomial in part (d) has two roots, -1 and 1, with 1 being a double root, and -1 being a triple root. The triple root "flattens" out the graph at the x-axis.

With the roots, we can sketch the graphs of each of these polynomials.

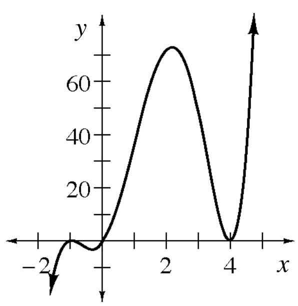
a.



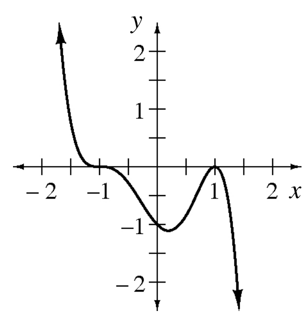
b.



c.



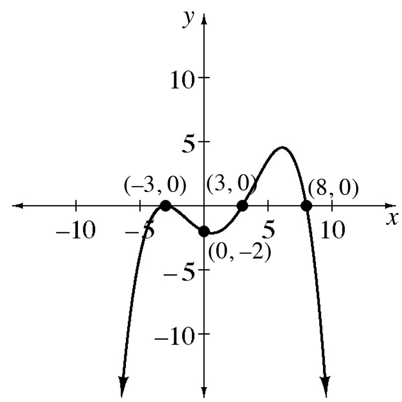
d.



Check that the roots fit the graphs. In addition, the graph in part (d) was the only one whose orientation was "flipped." Normally, a polynomial with an odd degree, starts off negative (as we move left of the graph) and ends up positive (as we move to the right). Because the polynomial in part (d) has a negative leading coefficient, its graph does the opposite.

Example 3

Write the exact equation of the graph shown at right.



From the graph we can write a general equation based on the orientation and the roots of the polynomial. Since the x-intercepts are -3, 3, and 8, we know $(x+3)$, $(x-3)$, and $(x-8)$ are factors. Also, since the graph touches at -3 and bounces off, $(x+3)$ is a double root, so we can write this function as $f(x)=a(x+3)^2(x-3)(x-8)$. We need to determine the value of a to have the exact equation.

Using the fact that the graph passes through the point $(0,-2)$, we can write:

$$-2 = a(0+3)^2(0-3)(0-8)$$
$$-2 = a(9)(-3)(-8)$$
$$-2 = 216a$$
$$a = -\tfrac{2}{216} = -\tfrac{1}{108}$$

Therefore the exact equation is $f(x) = -\frac{1}{108}(x+3)^2(x-3)(x-8)$.

Problems

State whether or not each of the following is a polynomial function. If it is, give the degree. If it is not, explain why not.

1. $\frac{1}{8}x^7 + 4.23x^6 - x^4 - \pi x^2 + \sqrt{2}x - 0.1$

2. $45x^3 - 0.75x^2 - \frac{3}{100}x + \frac{5}{x} + 15$

3. $x(x+2)\left(6+\frac{1}{x}\right)$

Sketch the graph of each of the following polynomials.

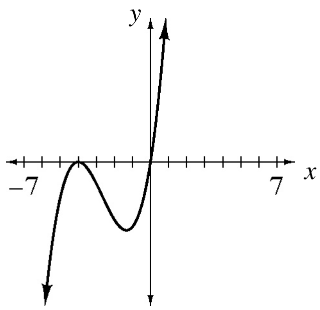
4. $y=(x+5)(x-1)^2(x-7)$

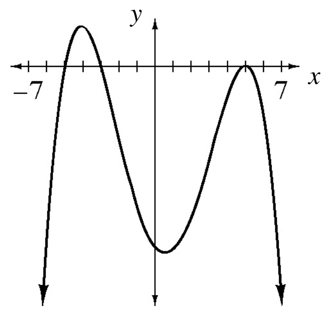
5. $y=-(x+3)(x^2+2)(x+5)^2$

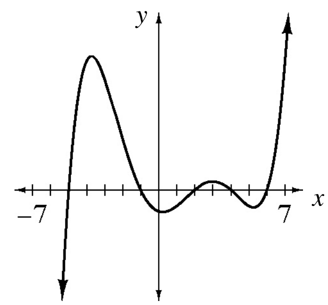
6. $f(x)=-x(x+8)(x+1)$

7. $y=x(x+4)(x^2-1)(x-4)$

Below are the complete graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots of the polynomial function, its degree, and orientation. Be sure to include information such as whether or not a root is a double or triple root.

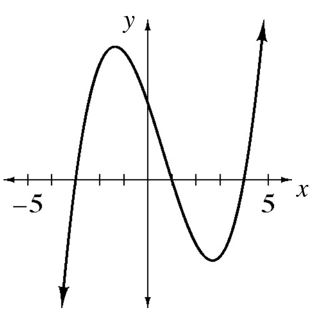
8. 

9. 

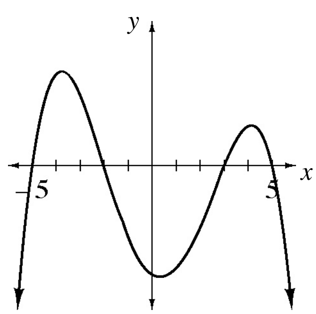
10. 

Using the graphs below and the given information, write the specific equation for each polynomial function.

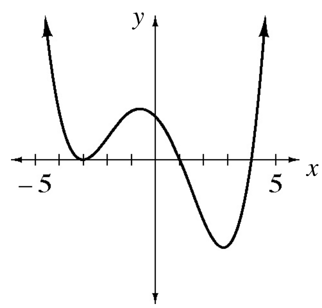
11. y-intercept: (0, 12)



12. y-intercept: (0, –15)

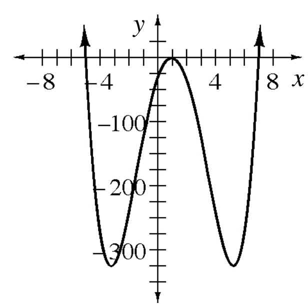


13. y-intercept: (0, 3)

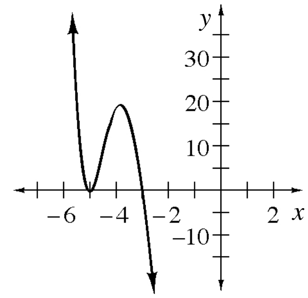


Answers

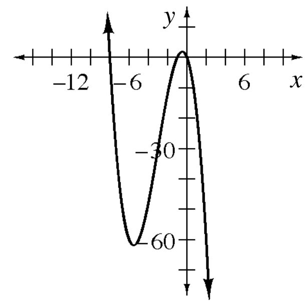
1. Yes, degree 7.
2. No. You cannot have x in the denominator.
3. No. When you multiply this out, you will still have x in the denominator.
4. The roots are $x = –5$, 1, and 7 with $x = 1$ being a double root. Remember a double root is where the graph is tangent. This graph has a positive orientation.



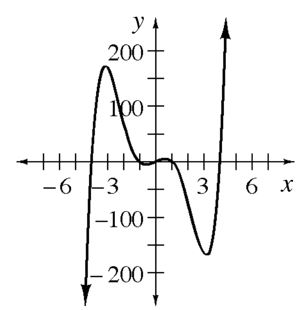
5. The roots are $x = -3$ and $x = -5$, which is a double root. The $x^2 + 2$ term does not produce any real roots since this expression cannot equal zero. The orientation is negative. The graph crosses the y-axis at $y = -150$.



6. This graph has negative orientation and the roots are $x = -8, -1$, and 0. Be sure to include $x = 0$ as a root.



7. $x^2 - 1$ gives us two roots. Since it factors to $(x + 1)(x - 1)$, the five roots are: $x = -4, -1, 0, 1$, and 4. The graph has a positive orientation.



8. A third degree polynomial (cubic) with one root at $x = 0$, and one double root at $x = -4$. It has a positive orientation.

9. A fourth degree polynomial with real roots at $x = -5$ and -3, and a double root at $x = 5$. It has a negative orientation.

10. A fifth degree polynomial with five real roots: $x = -5, -1, 2, 4$, and 6. It has a positive orientation.

11. $y = (x + 3)(x - 1)(x - 4)$

12. $y = -0.1(x + 5)(x + 2)(x - 3)(x - 5)$

13. $y = \frac{1}{12}(x+3)^2(x-1)(x-4)$

COMPLEX NUMBERS 8.2.1 and 8.2.3

Students are introduced to the complex number system. Complex numbers arise naturally when trying to solve some equations such as $x^2 + 1 = 0$, which, until now, students thought had no solution. They see how the solution to this equation relates to its graph, its roots, and how imaginary and complex numbers arise in other polynomial equations as well. For further information see the Math Notes boxes in Lessons 8.2.1, 8.2.2, and 8.2.3.

Example 1

Solve the equation below using the Quadratic Formula. Explain what the solution tells you about the graph of the function.

$$2x^2 - 20x + 53 = 0$$

As a quick review, the Quadratic Formula says: If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Here, $a = 2$, $b = -20$, and $c = 53$. Therefore,

$$\begin{aligned} x &= \frac{-(-20) \pm \sqrt{(-20)^2 - 4(2)(53)}}{2(2)} \\ &= \frac{20 \pm \sqrt{400 - 424}}{4} \\ &= \frac{20 \pm \sqrt{-24}}{4} \end{aligned}$$

We now have an expression with a negative under the radical. Until now, students would claim this equation has no solution. In fact, it has no **real** solution, but it does have a complex solution.

We define $i = \sqrt{-1}$ as an imaginary number. When we combine an imaginary number with a real number, we call it a complex number. Complex numbers are written in the form $a + bi$. Using i, we can simplify the answer above.

$$\begin{aligned} x &= \frac{20 \pm \sqrt{-24}}{4} \\ &= \frac{20 \pm \sqrt{-1 \cdot 4 \cdot 6}}{4} \\ &= \frac{20 \pm 2i\sqrt{6}}{4} \\ &= \frac{2\left(10 \pm i\sqrt{6}\right)}{4} \\ &= \frac{10 \pm i\sqrt{6}}{2} \end{aligned}$$

Because this equation has no real solutions, if we were to graph $y = 2x^2 - 20x + 53$ we would see a parabola that does not cross the x-axis. If we completed the square and put this into graphing form, we would get $y = 2(x - 5)^2 + 3$. The vertex of this parabola is at (5, 3), and since it open upwards, it will never cross the x-axis. You should verify this with your graphing tool.

Therefore, the graph of the function $y = 2x^2 - 20x + 53$ has no x-intercepts, but it does have two complex roots, $x = \frac{10 \pm i\sqrt{6}}{2}$. Recall that we said the degree of a polynomial function tells us the maximum number of roots. In fact the degree tells us the exact number of roots; some (or all) might be complex.

Example 2

Simplify each of the following expressions.

a. $3+\sqrt{-16}$

b. $(3+4i)+(-2-6i)$

c. $(4i)(-5i)$

d. $(8-3i)(8+3i)$

Remember that $i = \sqrt{-1}$. Therefore, the expression in (a) can be written as $3+\sqrt{-16} = 3+4\sqrt{-1} = 3+4i$. This is the simplest form; we cannot combine real and imaginary parts of the complex number. But, as is the case in part (b), we can combine real parts with real parts, and imaginary parts with imaginary parts: $(3 + 4i) + (-2 - 6i) = 1 - 2i$. In part (c), we can use the commutative rule to rearrange this expression: $(4i)(-5i) = (4 \cdot -5)(i \cdot i) = -20i^2$. However, remember that $i = \sqrt{-1}$, so $i^2 = (\sqrt{-1})^2 = -1$. Therefore, $-20i^2 = -20(-1) = 20$. Finally in part (d), we will multiply using methods we have used previously for multiplying binomials. You can use the Distributive Property or generic rectangles to compute this product.

$$\begin{aligned}(8-3i)(8+3i) &= 8(8)+8(3i)-3i(8)-3i(3i)\\ &= 64+24i-24i+9\\ &= 73\end{aligned}$$

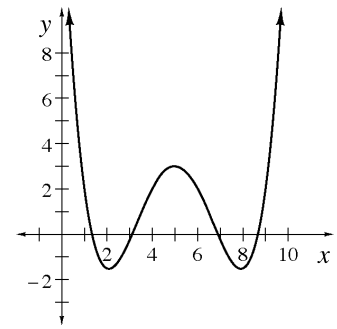
	8	$-3i$
8	64	$-24i$
$3i$	$24i$	9

The two expressions in part (d) are similar. In fact they are the same except for the middle sign. These two expressions are called **complex conjugates**, and they are useful when working with complex numbers. As you can see, multiplying a complex number by its conjugate produces a real number! This will always happen. Also, whenever a function with real coefficients has a complex root, it always has the conjugate as a root as well.

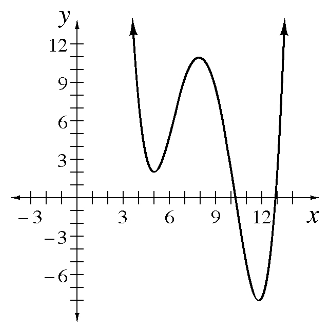
Example 3

Make a sketch of a graph of a polynomial function $p(x)$ so that $p(x) = 0$ would have only four real solutions. Change the graph so that it has two real and two complex solutions.

If $p(x) = 0$ is to have only four real solutions, then $p(x)$ will have four real roots. This will be a fourth degree polynomial that crosses the x-axis in exactly four different places. One such graph is shown at right.



In order for the graph to have only two real and two complex roots, we must change it so one of the "dips" does not reach the x-axis. One example is shown at right.



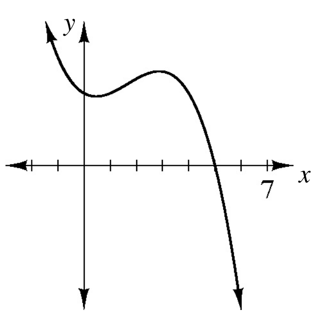
Problems

Simplify the following expressions.

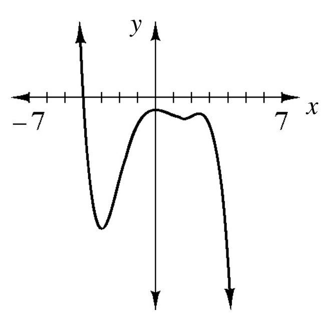
1. $(6 + 4i) - (2 - i)$
2. $8i - \sqrt{-16}$
3. $(-3)(4i)(7i)$
4. $(5 - 7i)(-2 + 3i)$
5. $(3 + 2i)(3 - 2i)$
6. $(\sqrt{3} - 5i)(\sqrt{3} + 5i)$

Below are the *complete* graphs of some polynomial functions. Based on the shape and location of the graph, describe all the roots of the polynomial function. Be sure to include information such as whether roots are double or triple, real or complex, etc.

7.



8.



9. Write the specific equation for the polynomial function passing through the point (0, 5), and with roots $x = 5$, $x = -2$ and $x = 3i$.

Answers

1. $4 + 5i$

2. $4i$

3. 84

4. $11 + 29i$

5. 13

6. 28

7. A third degree polynomial with negative orientation and with one real root at $x = 5$ and two complex roots.

8. A fifth degree polynomial with negative orientation and with one real root at $x = -4$ and four complex roots.

9. $y = -\frac{1}{18}(x^2 - 3x - 10)(x^2 + 9)$

FACTORING POLYNOMIAL FUNCTIONS 8.3.1 and 8.3.3

Students learn to divide polynomials as one method for factoring polynomials of degrees higher than two. Through division and with two theorems, students are able to rewrite polynomials in a form more suitable for graphing. They can also easily find a polynomial's roots, both real and complex. For further information see the Math Notes boxes in Lessons 8.3.1, 8.3.2, and 8.3.3.

Example 1

Divide $x^3 + 4x^2 - 7x - 10$ by $x + 1$.

Students have learned to multiply polynomials using several methods, one of which is with generic rectangles. The generic rectangle is a method that works for polynomial division as well.

To find the product of two polynomials, students draw a rectangle and label the dimensions with the two polynomials. The area of the rectangle is the product of the two polynomials. For division, we start with the area and one dimension of the rectangle, and use the model to find the other dimension.

To review, consider the product $(x + 2)(x^2 + 3x - 7)$. We use the two expressions as the dimensions of a rectangle and calculate the area of each smaller part of the rectangle. In this case, the upper left rectangle has an area of x^3. The next rectangle to the right has an area of $3x^2$. We continue to calculate each smaller rectangle's area, and sum the collection to find the total area. The total area represents the product.

	x^2	$3x$	-7
x	x^3	$3x^2$	$-7x$
2	$2x^2$	$6x$	-14

Here, the total area is $x^3 + 3x^2 - 7x + 2x^2 + 6x - 14$, or $x^3 + 5x^2 - x - 14$ once it is simplified.

Now we will do the reverse of this process for our example. We will set up a rectangle that has a width of $x + 1$ and an area of $x^3 + 4x^2 - 7x - 10$. We have to move slowly, however, as we do not know what the length will be. We will add information to the figure gradually, adjusting as we go. The top left rectangle has an area equal to the highest-powered term: x^3. Now work backwards: If the area of this rectangle is x^3 and the side has a length of x, what does the length of the other side have to be? It would be an x^2. If we fill this piece of information above the upper left small rectangle we can use it to compute the area of the lower left rectangle.

x	x^3	
1		

	x^2	
x	x^3	
1		

$\longrightarrow$

	x^2	
x	x^3	
1	x^2	

The total area is $x^3 + 4x^2 - 7x - 10$, but we only have $1x^3$ and $1x^2$ so far. We will need to add $3x^2$ more to the total area (plus some other terms, but remember we are taking this one step at a time).

Once we have filled in the remaining "x^2" area, we can figure out the length of the top side. Remember that part of the left side has a length of x. This means that part of the top must have a length of $3x$.

	x^2	$3x$	
x	x^3	$3x^2$	
1	x^2	$3x$	

Use this new piece of information to compute the area of the rectangle that is to the right of the x^3 rectangle, and then the small rectangle below that result.

Our total area has a total of $-7x$, but we have only $3x$ so far. This means we will need to add $-10x$ more. Place this amount of area in the rectangle to the right of $3x^2$.

With this new piece of area added, we can compute the top piece's length and use it to calculate the area of the rectangle below the $-10x$. Note that our constant term in the total area is -10, which is what our rectangle has as well.

	x^2	$3x$	-10
x	x^3	$3x^2$	$-10x$
1	x^2	$3x$	-10

Therefore we can write $\frac{x^3+4x^2-7x-10}{x+1} = x^2 + 3x - 10$, or $x^3 + 4x^2 - 7x - 10 = (x + 1)(x^2 + 3x - 10)$. Now that one of the terms is a quadratic, students can to factor it. Therefore, $x^3 + 4x^2 - 7x - 10 = (x + 1)(x + 5)(x - 2)$.

Example 2

Factor the polynomial and find all its roots.

$$P(x) = x^4 + x^2 - 14x - 48$$

Students learn the Integral Zero Theorem, which says that zeros, or roots of this polynomial, must be factors of the constant term. This means the possible real roots of this polynomial are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24$, or ±48. In this case there are 20 possible roots to check! We can check them in a number of ways. One method is to divide the polynomial by the corresponding binomial expression (for instance, if -1 is a root, we divide the polynomial by $(x + 1)$ to see if it is a factor. Another method is to substitute each zero into the polynomial to see which of them, if any, make the polynomial equal to zero. We will still have to divide by the corresponding expression once we have the root, but it will mean less division in the long run.

Substituting values for $P(x)$, we get:

$$\begin{aligned} P(1) &= (1)^4 + (1)^2 - 14(1) - 48 \\ &= 1 + 1 - 14 - 48 \\ &= -60 \end{aligned}$$

$$\begin{aligned} P(-1) &= (-1)^4 + (-1)^2 - 14(-1) - 48 \\ &= 1 + 1 + 14 - 48 \\ &= -32 \end{aligned}$$

$$\begin{aligned} P(2) &= (2)^4 + (2)^2 - 14(2) - 48 \\ &= 16 + 4 - 28 - 48 \\ &= -56 \end{aligned}$$

$$\begin{aligned} P(-2) &= (-2)^4 + (-2)^2 - 14(-2) - 48 \\ &= 16 + 4 + 28 - 48 \\ &= 0 \end{aligned}$$

We can keep going, but we just found a root, $x = -2$. Therefore, $x + 2$ is a factor of the polynomial. Now we can divide the polynomial by this factor to find the other factors.

	x^3	$-2x^2$	$5x$	-24
x	x^4	$-2x^3$	$5x^2$	$-24x$
2	$2x^3$	$-4x^2$	$10x$	-48

This other factor, however, is degree three, still too high to use easier methods of factoring. Therefore we must use the Integral Zero Theorem again, and find another zero from the list ±1, $\pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$. We can start where we left off, but now using $Q(x) = x^3 - 2x^2 + 5x - 24$, a simpler polynomial to evaluate.

$$\begin{aligned} Q(3) &= (3)^3 - 2(3)^2 + 5(3) - 24 \\ &= 27 - 18 + 15 - 24 \\ &= 0 \end{aligned}$$

It is certainly helpful to find a root so quickly! Since $x = 3$ is a root, $x - 3$ is a factor. Therefore, we can divide again.

	x^2	x	8
x	x^3	x^2	$8x$
-3	$-3x^2$	$-3x$	-24

$$x^2 + x + 8 = 0$$

$$\begin{aligned} x &= \tfrac{-1 \pm \sqrt{1^2 - 4(1)(8)}}{2(1)} \\ &= \tfrac{-1 \pm \sqrt{1-32}}{2} \\ &= \tfrac{-1 \pm \sqrt{-31}}{2} \\ &= \tfrac{-1 \pm i\sqrt{31}}{2} \end{aligned}$$

Now we have $P(x) = x^4 + x^2 - 14x - 48 = (x + 2)(x - 3)(x^2 + x + 8)$. Finally! The last polynomial is a quadratic (degree 2) so we can factor or use the Quadratic Formula. If you try factoring, you will not be successful, as this quadratic does not factor with integers. Therefore, we must use the Quadratic Formula to find the roots as shown at right.

Therefore, the original polynomial factors as:

$$P(x) = x^4 + x^2 - 14x - 48 = (x+2)(x-3)\left(x - \tfrac{-1+i\sqrt{31}}{2}\right)\left(x - \tfrac{-1-i\sqrt{31}}{2}\right)$$

Problems

1. Divide $3x^3 - 5x^2 - 34x + 24$ by $3x - 2$.
2. Divide $x^3 + x^2 - 5x + 3$ by $x - 1$.
3. Divide $6x^3 - 5x^2 + 5x - 2$ by $2x - 1$.

Factor the polynomials, keeping the factors real.

4. $f(x) = 2x^3 + x^2 - 19x + 36$
5. $g(x) = x^4 - x^3 - 11x^2 - 5x + 4$

Find all roots for each of the following polynomials.

6. $P(x) = x^4 - 2x^3 + x^2 - 8x - 12$
7. $Q(x) = x^3 - 14x^2 + 65x - 102$

Answers

1. $x^2 - x - 12$
2. $x^2 + 2x - 3$
3. $3x^2 - x + 2$
4. $f(x) = (x + 4)(2x^2 - 7x + 9)$
5. $g(x) = (x + 1)(x - 4)(x^2 + 2x - 1)$
6. $x = -1, 3, 2i, -2i$
7. $x = 6, 4 + i, 4 - i$

SAT PREP

1. $(5 + 6)^2 = ?$

   a. $(2 \cdot 5) + (2 \cdot 6)$ b. $5^2 + 6^2$ c. 11^2 d. 61 e. $5^2 \times 6^2$

2. If $6x - 7y = 12$, what is the value of $-2(6x - 7y)$?

   a. –24 b. 13 c. –1 d. 420 e. –42

3. The average (arithmetic mean) of three numbers is 25. If two numbers are 25 and 30, what is the third number?

   a. 35 b. 30 c. 25 d. 20 e. 15

4. People from the country of Turpa measure with different units. Each curd is 7 garlongs long and each garlong is made up of 15 bleebs. How many complete curds are there in 510 bleebs?

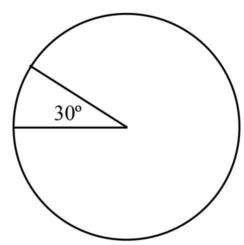
   a. 105 b. 15 c. 5 d. 4 e. 2

5. If $x^2 - y^2 = 12$ and $x - y = 2$, what is the value of $x + y$?

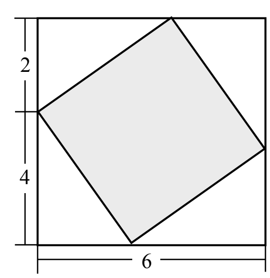
6. Five consecutive integers sum to 25. What is the largest of these consecutive numbers?

7. For all positive integers m and n, we define $m \nearrow n$ to be the whole number remainder when m is divided by n. If $11 \nearrow k = 3$, what does k equal?

8. At Pies R Us, each pie is cut into a slice as shown in the figure at right. Each slice of pie has a central angle of 30º. They sell the pies by the slice. If the weight of each pie is uniformly distributed, weighing 108 grams, how much does each slice weigh in grams?



9. In the figure at right, what is the area of the shaded region if that region is a square?



10. What is the sixth term in the sequence beginning 432, 72, 12, … ?

Answers

1. C	2. A	3. D	4. D	5. 6
6. 7	7. 8	8. 9 g	9. 20	10. $\frac{1}{18}$

SAT PREP

1. $(5 + 6)^2 = ?$

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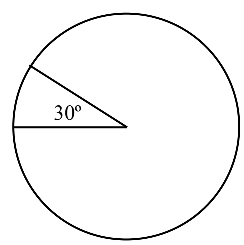

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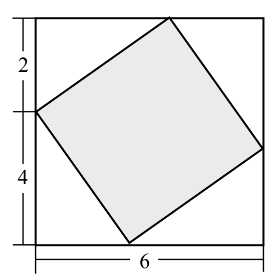

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